Some Conditions for Convexifying Static H_∞ Control Problems ⋆

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Abstract: In this paper, we show that a set of static controllers satisfying a certain level of H_∞ performance becomes convex when the underlying generalized plant satisfy several structural conditions. More precisely, we characterize such static H_∞ controllers by an LMI with the controller parameters being kept directly as decision variables. The conditions on the generalized plant are not too strict as illustrated by the fact that a sort of mixed sensitivity problems indeed satisfies these conditions. In addition, for the generalized plant of interest, we prove that full-order dynamical H_∞ controllers can be characterized by an LMI with simple change of variables. In stark contrast to known LMI formulations, the change of variables does not involve coefficient matrices of the generalized plant. This property is promising when dealing with a whole variety of robust control problems. As an illustration, the real μ synthesis problem is discussed.

Keywords: Convex optimization, H_∞ control, linear matrix inequalities (LMIs).

1. INTRODUCTION

Recently, convex optimization has been a standard strategy for control system analysis and synthesis. In particular, linear matrix inequality (LMI) and semidefinite programming (SDP) are widely accepted with the help of freely available powerful softwares. In retrospect, one of the major reasons why LMI attracted such intensive attention would be the fact that the H_∞ control problem, the central issue of the robust control theory, has been solved completely by means of LMIs. Prominent result is the elimination of controller variable approach independently conceived by Gahinet and Apkarian (1994) and Iwasaki and Skelton (1994). Subsequently, Scherer et al. (1997) and Masubuchi et al. (1998) independently proposed the linearizing change of controller variable approach. These pioneering works are then extended to other control problems such as gain-scheduled controller synthesis.

Even though these works opened up a new horizon for the H_∞ control theory, one possible criticism is that these approaches do not provide LMIs that keep controller variables directly as decision variables, and in particular, the controller variables are characterized as a function of plant data. This surely restricts the scope of their application. For example, we cannot deal with structural constraints on the controller in a straightforward fashion.

In this paper, we show that a set of static controllers satisfying a certain level of H_∞ performance becomes convex when the underlying generalized plant satisfy several conditions. More precisely, we characterize such static H_∞ controllers by an LMI with the controller parameters being kept directly as decision variables. Even in the case of dynamical controller synthesis, we can conceive novel convexity results for the generalized plant of interest. Indeed, if we focus on the full-order dynamical H_∞ controller synthesis, it turns out that such full-order controller can be characterized by an LMI with simple change of variables. In stark contrast to (Scherer et al., 1997; Masubuchi et al., 1998), the change of variables in the present paper does not involve plant data and this property is promising when dealing with a whole variety of robust control problems.

We note that the static controller synthesis problem, which is nonconvex in general, has been studied intensively in Henrion and Lasserre (2005, 2006) from a broad perspective. Differently from these real algebraic geometry approaches, the basic spirit of the present study is overcoming nonconvexity by exploiting specific structures of the H_∞ control problem.

We use the following notations. For $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, we define $\mathrm{He}\{A\} := A + A^T$ and $\mathrm{Sq}\{B\} = BB^T$. The set of positive definite matrices of the size n are denoted by $\mathbf{S}^n_+$. Others notations are standard.

2. CONVEXITY OF STATIC H_∞ CONTROLLERS

In this section, we show that a set of static controllers satisfying a certain level of H_∞ performance becomes convex under several conditions on the generalized plant. A concrete example of the generalized plant satisfying required conditions is also given.

⋆ This work is supported in part by the Ministry of Education, Culture, Sports, Science and Technology of Japan under Grant-in-Aid for Young Scientists (B), 21760328.

2.1 Structure of Generalized Plant Ensuring Convexity

Let us consider the generalized plant G_γ depicted in Fig. 1, where K denotes the controller to be designed. Suppose the state space realization of G_γ is given by

$$G_\gamma : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, \\ z = C_{1,\gamma} x + D_{11} w + D_{12} u, \\ y = C_2 x + D_{21} w. \end{cases} \tag{1}$$

Here, $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_w}$ the disturbance input, $u \in \mathbf{R}^{m_u}$ the control input, $z \in \mathbf{R}^{l_z}$ the performance output, and $y \in \mathbf{R}^{l_y}$ the measured output, respectively. We consider the case where C_1 is a continuous function of $\gamma > 0$ as in $C_{1,\gamma}$, where γ stands for the H_∞ performance level to be minimized.

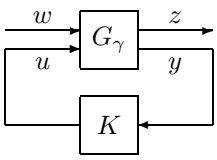

Fig. 1. Generalized plant for H_∞ controller synthesis.

For a given controller K, static or dynamic, let us denote by $T(G_\gamma, K)$ the closed-loop system as in Fig. 1, where the input and the output are w and z, respectively. Moreover, for static controllers $K \in \mathbb{R}^{m_u \times l_y}$, we define the set

$$\mathcal{K}_\gamma := \left\{ K : \ K \in \mathbb{R}^{m_u \times l_y}, \ \|T(G_\gamma, K)\|_\infty < 1 \right\}. \tag{2}$$

Our main concern in this paper is under what condition on G_γ the set $\mathcal{K}_\gamma$ defined by (2) becomes convex. It turns out that the following assumption will suffice.

Assumption 1.

(i) The matrix A is Hurwitz stable.
(ii) $D_{11} = 0$ and $D_{21} = 0$.
(iii) $C_{1,\gamma}^T D_{12} = 0$.
(iv) The matrix B_1 can be partitioned as $B_1 = [\ B_2 \ B_{12}\ ]$ for some matrix B_{12}.
(v) $D_{12}^T D_{12} \succeq I_{m_u}$.

Under these preparations, we can establish Theorem 1 given below. For ease of description, here we define

$$M_\gamma := \{A, B_1, B_2, C_{1,\gamma}, D_{11}, D_{12}, C_2\}.$$

Theorem 1. For given $\gamma > 0$ and the generalized plant G_γ satisfying Assumption 1, the set of static controllers $\mathcal{K}_\gamma$ defined by (2) is convex if it is not empty. In particular, the set $\mathcal{K}_\gamma$ can be characterized by an LMI as follows:

$$\begin{aligned} \mathcal{K}_\gamma = \big\{ K : \ & K \in \mathbf{R}^{m_u \times l_y}, \\ & \exists P \in \mathbf{S}_+^n \text{ such that } L(M_\gamma, P, K) \prec 0 \big\}. \end{aligned} \tag{3}$$

Here, $L(M_\gamma, P, K)$ is given (4) at the top of the next page.

Proof. Let us denote by $(A_{\rm cl}, B_{\rm cl}, C_{\rm cl}, D_{\rm cl})$ the state space matrices of the closed-loop system $T(G_\gamma, K)$. They can be represented by

$$\begin{aligned} A_{\rm cl} &= A + B_2 K C_2, \quad B_{\rm cl} = B_1, \\ C_{\rm cl} &= C_1 + D_{12} K C_2, \ D_{\rm cl} = 0 \end{aligned} \tag{5}$$

where the assumption (ii) is used tacitly. Then, from the bounded real lemma (Boyd et al., 1994), we see that $\|T(G_\gamma, K)\| < 1$ holds if and only if there exists $P \in \mathbf{S}_+^n$ such that

$$\mathrm{He}\{PA_{\rm cl}\} + \mathrm{Sq}\{PB_{\rm cl}\} + \mathrm{Sq}\{C_{\rm cl}^T\} \prec 0.$$

From (5) and the assumption (iv), this inequality can be rewritten equivalently as

$$\begin{aligned} &\mathrm{He}\{PA\} + \mathrm{Sq}\{PB_2 + C_2^T K^T\} + \mathrm{Sq}\{PB_{12}\} \\ &\quad + \mathrm{Sq}\{C_{1,\gamma}^T\} + C_2^T K^T (D_{12}^T D_{12} - I) K C_2 \prec 0. \end{aligned}$$

where we also used the assumption (iii). Since $D_{12}^T D_{12} - I \succeq 0$ from (v), we can rewrite the above inequality as in $L(M_\gamma, P, K) \prec 0$ via Schur complement, where $L(M_\gamma, P, K)$ is given in (4). It is obvious that $L(M_\gamma, P, K) \prec 0$ holds only if the matrix A is Hurwitz stable and this justifies the inclusion of (i). □

Recently, convex optimization has been a standard tool for control system analysis and synthesis. Therefore, convexity of the set of controllers satisfying a certain performance is an issue of great interest. Nevertheless, it is a well-known fact that the very basic set, i.e., the set of static stabilizing controllers is nonconvex and even disconnected in general. Due to this fact, to the best of the author's knowledge, the study seeking for convexity of static H_∞ controllers is rare in the literature. We admit that Assumption 1 might be rather stringent, but viewed from a different angle, the resulting convexity is very useful when dealing with a whole variety of control problems. The LMI-based systematic synthesis of static H_∞ controllers we have just established in Theorem 1 is a typical example. As is well-known, this is beyond reach in general.

2.2 Concrete Example of Generalized Plant

In this subsection, we give a concrete example of the generalized plant satisfying Assumption 1.

Let us consider the mixed sensitivity problem for the plant P described by

$$P(s) = \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & 0 \end{array} \right].$$

We assume that P is an SISO system just for simplicity. As usual, we assume that weighting functions W_S and W_T are appropriately designed for the shaping of sensitivity and complementary sensitivity functions, respectively. Since W_S and W_T are typically chosen to be low-pass and high-pass, suppose their state space realizations are given by

$$W_S(s) = \left[\begin{array}{c|c} A_S & B_S \\ \hline C_S & 0 \end{array} \right], \ W_T(s) = \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array} \right].$$

Here, the matrices A_S and A_T are Hurwitz stable.

If we place these weighting functions on the input-side of P, a block-diagram for the mixed sensitivity problem can be represented as Fig. 2. Then, one of the standard settings for the mixed sensitivity problem will be

$$\inf_K \quad \gamma \quad \text{subject to } \|T(G_\gamma, K)\|_\infty < 1 \tag{6}$$

where G_γ can be written explicitly as

$$L(M_\gamma, P, K) := \begin{bmatrix} \mathrm{He}\{PA\} + \mathrm{Sq}\{C_{1,\gamma}^T\} & PB_2 + C_2^T K^T & PB_{12} & C_2^T K^T (D_{12}^T D_{12} - I)^{1/2} \\ * & -I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} \tag{4}$$

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$$G_\gamma(s) = \left[\begin{array}{ccc|c:c} A_P & 0 & 0 & B_P & B_P \\ 0 & A_S & 0 & B_S & B_S \\ 0 & 0 & A_T & 0 & B_T \\ \hline 0 & \frac{1}{\gamma}C_S & 0 & 0 & 0 \\ 0 & 0 & C_T & 0 & D_T \\ \hdashline C_P & 0 & 0 & 0 & 0 \end{array}\right]. \tag{7}$$

Unfortunately, as expected, this generalized plant does not satisfy Assumption 1 unless $B_T = 0$ and $D_T \geq 1$.

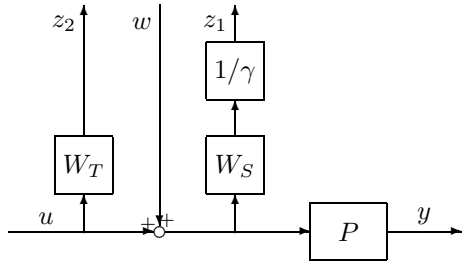

Fig. 2. Standard generalized plant for mixed sensitivity problem with weightings on input-side.

In view of these facts, we try to modify Fig. 2 while preserving the basic spirit of sensitivity and complementary sensitivity function shaping via W_S and W_T. One of the possible modifications turns out to be Fig. 3, where the state space realization of the corresponding G_γ is

$$G_\gamma(s) = \left[\begin{array}{ccc|cc:c} A_P & 0 & B_P C_T & B_P & B_P D_T & B_P \\ 0 & A_S & 0 & B_S & 0 & B_S \\ 0 & 0 & A_T & 0 & B_T & 0 \\ \hline 0 & \frac{1}{\gamma}C_S & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline C_P & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{8}$$

We can confirm that this generalized plant certainly satisfies Assumption 1 if A_P is Hurwitz stable.

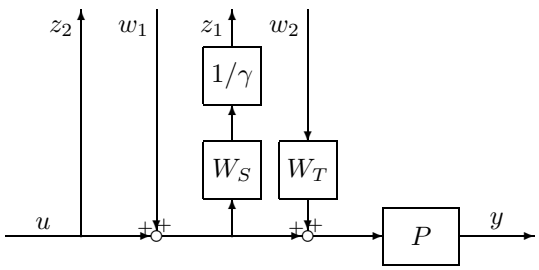

Fig. 3. Modified generalized plant for Fig. 2.

It is of course important to compare the two strategies of the H_∞ controller synthesis based on the generalized plants in Figs. 2 and 3. If we employ Fig. 2, the constraint $\|T(G_\gamma, K)\|_\infty < 1$ can be written explicitly as

$$\left\| \begin{bmatrix} \frac{1}{\gamma}W_S S \\ -W_T T \end{bmatrix} \right\|_\infty < 1 \tag{9}$$

where $S := (1 + KP)^{-1}$ and $T := (1 + KP)^{-1}KP$. On the other hand, in the case of Fig. 3, the constraint $\|T(G_\gamma, K)\|_\infty < 1$ becomes

$$\left\| \begin{bmatrix} \frac{1}{\gamma}W_S S & -\frac{1}{\gamma}W_S T W_T \\ -T & -T W_T \end{bmatrix} \right\|_\infty < 1 \tag{10}$$

We have the following remarks regarding the comparison between (9) and (10).

(a) The constraint (10) implies $\|\frac{1}{\gamma}W_S S\|_\infty < 1$ and $\|TW_T\|_\infty < 1$. In this sense, the shaping of S and T via W_S and W_T will be achieved.

(b) The constraint (10) implies $\|T\|_\infty < 1$, i.e., it restricts the H_∞ norm of the unweighted complementary sensitivity function under unity. In the well-designed feedback control systems, the frequency response $T(j\omega)$ typically satisfies $\|T(j\omega)\| \simeq 1$ at low frequency range and $\|T(j\omega)\| \ll 1$ at high frequency range. Therefore, the restriction $\|T(j\omega)\| < 1$ would not be stringent in these frequency ranges. However, the restriction $\|T\|_\infty < 1$ can be a source of conservatism when we want to improve the frequency response of overall system at middle frequency range.

(c) The constraint (10) implies $\|\frac{1}{\gamma}W_S T W_T\|_\infty < 1$ and again this might be a source of conservatism. However, the effect of this constraint can be reduced if $\|W_S W_T\|_\infty \ll 1$. We can expect that this condition is usually satisfied to some extent, since W_S and W_T are typically chosen as low-pass and high-pass, respectively.

(d) Since $\frac{1}{\gamma}W_S S$ and TW_T enter into the diagonal blocks in (10), we cannot draw any definite conclusion on the inclusion relationship among the two sets K_γ corresponding to the generalized plants in Fig. 2 and Fig. 3. However, from numerical examples, it is observed that the modification from Fig. 2 to Fig. 3 shrinks K_γ in most cases, possibly due to the reasons stated in (b) and (c).

(e) We note that the LMI $L(M_\gamma, P, K) \prec 0$ in (3) holds *only if* A_P is Hurwitz stable. In other words, the set K_γ is empty if A_P is not stable. This can be interpreted that the stabilization problem has been excluded from our scope so that the convexity of the static controllers can be ensured.

2.3 Numerical Example

Suppose the plant P is given by

$$P(s) = \frac{K_P \beta \omega_n^2}{(s+\beta)(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{11}$$

where $K_P = 1$, $\beta = 0.1$, $\zeta = 0.1$ and $\omega_n = 1$. For this plant, we construct the generalize plant in Fig. 3 by letting W_S and W_T as follows:

$$W_S(s) = \frac{1}{(1+10s)^2}, \quad W_T(s) = \frac{0.05+5s}{1+0.2s}. \tag{12}$$

As we have already shown, the state space realization of the corresponding G_γ given in (8) satisfies Assumption 1. Therefore, we can compute the H_∞ optimal static gain for the mixed sensitivity problem (6) by solving the SDP:

$$\inf_{K,P} \quad \gamma \quad \text{subject to } L(M_\gamma, P, K) \prec 0.$$

It turns out that the optimal static controller is $K_{\rm opt} = -0.3336$, achieving the H_∞ performance $\gamma_{\rm opt}^{\rm static} = 0.77$.

3. DYNAMICAL H_∞ CONTROLLER SYNTHESIS

Let us move on to the synthesis of dynamical controllers of the form

$$K: \begin{cases} \dot{x}_c = A_c x_c + B_c y, \\ u = C_c x_c + D_c y \end{cases} \tag{13}$$

where $x_c \in \mathbf{R}^{n_c}$. As in the preceding section, we are interested in whether the set of controllers K satisfying $\|T(G_\gamma, K)\| < 1$ becomes convex in the parameters (A_c, B_c, C_c, D_c) under Assumption 1. Unfortunately, this seems too demanding and beyond reach as explicated later on. However, it turns out that Assumption 1 still brings novel convexity results for dynamical controller synthesis as well.

In the sequel, we assume that the matrix A in (1) is Hurwitz stable.

3.1 Convexity of (C_c, D_c) for fixed (A_c, B_c)

By following the standard procedure for dynamical controller synthesis, let us first write the state space matrices $(A_{\rm cl}, B_{\rm cl}, C_{\rm cl}, D_{\rm cl})$ of the closed-loop system $T(G_\gamma, K)$ as

$$\begin{aligned} A_{\rm cl} &= \widetilde{\mathsf{A}} + \widetilde{\mathsf{B}}_2 \widetilde{\mathsf{K}} \widetilde{\mathsf{C}}_2, \quad B_{\rm cl} = \widetilde{\mathsf{B}}_1, \\ C_{\rm cl} &= \widetilde{\mathsf{C}}_1 + \widetilde{\mathsf{D}}_{12} \widetilde{\mathsf{K}} \widetilde{\mathsf{C}}_2, \ D_{\rm cl} = 0. \end{aligned} \tag{14}$$

Here, we defined

$$\begin{aligned} &\widetilde{\mathsf{A}} := \begin{bmatrix} A & 0 \\ 0 & 0_{n_c} \end{bmatrix}, \ \widetilde{\mathsf{B}}_1 := \begin{bmatrix} B_1 \\ 0_{n_c, m_w} \end{bmatrix}, \\ &\widetilde{\mathsf{B}}_2 := \begin{bmatrix} B_2 & 0 \\ 0 & I_{n_c} \end{bmatrix}, \ \widetilde{\mathsf{C}}_{1,\gamma} := [\, C_{1,\gamma} \ 0_{l_z, n_c} \,], \\ &\widetilde{\mathsf{D}}_{12} := [\, D_{12} \ 0_{l_z, n_c} \,], \ \widetilde{\mathsf{C}}_2 := \begin{bmatrix} C_2 & 0 \\ 0 & I_{n_c} \end{bmatrix}, \ \widetilde{\mathsf{K}} := \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}. \end{aligned} \tag{15}$$

Furthermore, let us define

$$\widetilde{G}_\gamma(s) := \left[\begin{array}{c|cc} \widetilde{\mathsf{A}} & \widetilde{\mathsf{B}}_1 & \widetilde{\mathsf{B}}_2 \\ \hline \widetilde{\mathsf{C}}_{1,\gamma} & 0 & \widetilde{\mathsf{D}}_{12} \\ \widetilde{\mathsf{C}}_2 & 0 & 0 \end{array}\right]. \tag{16}$$

As is well-known, this procedure enables us to deal with the dynamical controller synthesis problem as if static controller synthesis problem since $T(G_\gamma, K) = T(\widetilde{G}_\gamma, \widetilde{\mathsf{K}})$ holds. In particular, since the structure of (14) conforms to (5), we can conclude that the set $\mathcal{K}_\gamma^{\rm dy}$, defined by

$$\mathcal{K}_\gamma^{\rm dy} := \left\{ \widetilde{\mathsf{K}} : \ \widetilde{\mathsf{K}} \in \mathbf{R}^{(n_c+m_u)\times(n_c+l_y)}, \|T(\widetilde{G}_\gamma, \widetilde{\mathsf{K}})\|_\infty < 1 \right\}, \tag{17}$$

becomes convex if the state space matrices in (16) satisfy the conditions in Assumption 1. Unfortunately, however, it is obvious from (15) that (i), (iv), and (v) are never satisfied unless we let $n_c = 0$.

To examine in more detail what convexity results can be obtained under Assumption 1, let us focus on the alternative representation of $(A_{\rm cl}, B_{\rm cl}, C_{\rm cl}, D_{\rm cl})$ given by

$$\begin{aligned} A_{\rm cl} &= \mathsf{A}(A_c, B_c) + \mathsf{B}_2 \mathsf{K} \mathsf{C}_2, \ B_{\rm cl} = \mathsf{B}_1, \\ C_{\rm cl} &= \mathsf{C}_{1,\gamma} + \mathsf{D}_{12} \mathsf{K} \mathsf{C}_2, \qquad D_{\rm cl} = 0, \end{aligned} \tag{18}$$

Here, we defined

$$\begin{aligned} &\mathsf{A}(A_c, B_c) := \begin{bmatrix} A & 0 \\ B_c C_2 & A_c \end{bmatrix}, \ \mathsf{B}_1 := \begin{bmatrix} B_1 \\ 0_{n_c, m_w} \end{bmatrix}, \\ &\mathsf{B}_2 := \begin{bmatrix} B_2 \\ 0_{n_c, m_u} \end{bmatrix}, \ \mathsf{C}_{1,\gamma} := [\, C_{1,\gamma} \ 0_{l_z, n_c} \,], \\ &\mathsf{D}_{12} := D_{12}, \ \mathsf{C}_2 := \begin{bmatrix} C_2 & 0 \\ 0 & I_{n_c} \end{bmatrix}, \ \mathsf{K} := [\, D_c \ C_c \,]. \end{aligned} \tag{19}$$

If we fix the matrices A_c and B_c and incorporate them into the plant side, we can confirm that $T(G_\gamma, K) = T(G_\gamma(A_c, B_c), \mathsf{K})$ holds where

$$G_\gamma(A_c, B_c, s) := \left[\begin{array}{c|cc} \mathsf{A}(A_c, B_c) & \mathsf{B}_1 & \mathsf{B}_2 \\ \hline \mathsf{C}_{1,\gamma} & 0 & \mathsf{D}_{12} \\ \mathsf{C}_2 & 0 & 0 \end{array}\right]. \tag{20}$$

Moreover, it is straightforward to see that the above state space matrices satisfy Assumption 1 if we let A_c to be Hurwitz stable. It follows that the set of admissible $\mathsf{K} = [\, C_c \ D_c \,]$ becomes convex for fixed A_c and B_c. This result can be stated formally as in the next theorem.

Theorem 2. For given $\gamma > 0$ and the generalized plant G_γ satisfying Assumption 1, let us consider the synthesis of dynamical controller K of the form (13). Then, for each fixed (A_c, B_c) with A_c being Hurwitz stable, the set $\mathcal{K}_\gamma(A_c, B_c)$ defined by

$$\begin{aligned} \mathcal{K}_\gamma(A_c, B_c) := \Big\{ \mathsf{K} : \ \mathsf{K} = [\, D_c \ C_c \,] \in \mathbf{R}^{m_u \times (l_y + n_c)}, \\ \|T(G_\gamma(A_c, B_c), \mathsf{K})\|_\infty < 1 \Big\} \end{aligned} \tag{21}$$

is convex if it is not empty. In particular, the set $\mathcal{K}_\gamma(A_c, B_c)$ can be characterized by an LMI as follows:

$$\begin{aligned} \mathcal{K}_\gamma(A_c, B_c) = \Big\{ \mathsf{K} : \ \mathsf{K} = [\, D_c \ C_c \,] \in \mathbf{R}^{m_u \times (l_y + n_c)}, \\ \exists P \in \mathbf{S}_+^{n+n_c} \text{ such that } L(\mathsf{M}_\gamma(A_c, B_c), P, \mathsf{K}) \prec 0 \Big\}. \end{aligned} \tag{22}$$

Here, $\mathsf{M}_\gamma(A_c, B_c)$ is defined through (19) as $\mathsf{M}_\gamma(A_c, B_c) := \{\mathsf{A}(A_c, B_c), \mathsf{B}_1, \mathsf{B}_2, \mathsf{C}_{1,\gamma}, \mathsf{D}_{12}, \mathsf{C}_2\}$.

In view of the results in Theorems 1 and 2, we could say that Assumption 1 has the effect that it convexifies (C_c, D_c) of the controller to be designed.

Theorem 2 clarifies that the H_∞ control problem of interest requires *strong stabilization*. Namely, the condition $\|T(G_\gamma, K)\|_\infty < 1$ holds *only if* K is internally stable, i.e., the matrix A_c in (13) is Hurwitz stable. This can be readily seen if we note that the LMI $L(\mathsf{M}_\gamma(A_c, B_c), P, \mathsf{K}) \prec 0$ contains Lyapunov inequality with respect to the matrix A_c. Therefore, the fixation of A_c to a stable matrix is not restrictive as long as we try to satisfy $\|T(G_\gamma, K)\|_\infty < 1$ for the generalized plant G_γ satisfying Assumption 1. The convex characterization in Theorem 2 becomes indeed

effective in conjunction with a finite-dimensional approximation of $K(s) \in RH_\infty$ (Hindi et al., 1998; Scherer, 2000). Details are omitted due to limited space.

3.2 Full-Order Controller Synthesis via New Change of Variables

In the preceding subsection, we have shown an LMI-based strategy for dynamical H_∞ controller synthesis of any order. The controller variables (C_c, D_c) are kept directly as LMI variables at the expense of the fixation of (A_c, B_c). However, it is nonetheless useful if we can directly optimize (A_c, B_c) as well in most problem instances.

In this subsection, we consider full-order controller synthesis ($n_c = n$ in (13)) and show that such direct optimization of (A_c, B_c) is indeed possible, provided that we allow them being involved in a linearizing change of variables. It should be noted that, if we give up the idea of deriving LMI conditions that keep controller variables directly as decision variables, the LMI-based H_∞ controller characterization has been done completely in the literature. However, the present approach still has its own benefits in comparison with known LMI formulations mainly because of the following reasons:

(i) contrary to (Gahinet and Apkarian, 1994; Iwasaki and Skelton, 1994), the controller variables are kept as decision variables without being eliminated;
(ii) contrary to (Scherer et al., 1997; Masubuchi et al., 1998), the linearizing change of variables in the present paper is very simple and does not involve the state space matrices of the generalized plant in the controller parametrization.

We emphasize that these properties are particularly useful when we deal with robust controller synthesis problems against parametric uncertainties (Barmish, 1994; Boyd et al., 1994). Roughly speaking, in the present approach, we can deal with dynamical controller synthesis as if static state-feedback controller synthesis via well-known simple change of variable (Scherer et al., 1997). The rest of this section is devoted to the technical details to verify the assertions stated above.

In order to derive the desired LMI condition, let us revisit the matrix inequality condition

$$L(\mathsf{M}_\gamma(A_c, B_c), P, \mathsf{K}) \prec 0 \tag{23}$$

which is presented in (22). If we consider (A_c, B_c) as decision variables as well, this matrix inequality condition is a BMI since a bilinear term appears among (A_c, B_c) and $P \in \mathbf{S}_+^{2n}$. To get around this difficulty, let us first consider partitioning P. Due to the freedom of the similarity transformation of the controller, it is shown in Masubuchi et al. (1998) that we can let P as follows without introducing any conservatism:

$$P = \begin{bmatrix} X & Z \\ Z & Z \end{bmatrix}, \quad X, Z \in \mathbf{S}_+^n. \tag{24}$$

Then, the sole bilinear term in (23) given by $P\mathsf{A}(A_c, B_c)$ can be linearized as

$$\begin{bmatrix} X & Z \\ Z & Z \end{bmatrix} \begin{bmatrix} A & 0 \\ B_c C_2 & A_c \end{bmatrix} \leftrightarrow \begin{bmatrix} XA + Y_{B_c} C_2 & Y_{A_c} \\ ZA + Y_{B_c} C_2 & Y_{A_c} \end{bmatrix}. \tag{25}$$

Here, we introduced the linearizing change of variables

$$Y_{A_c} := ZA_c, \quad Y_{B_c} := ZB_c. \tag{26}$$

It follows that the BMI condition (23) can be reduced into the LMI condition of the following form:

$$\widehat{L}(M_\gamma, X, Z, Y_{A_c}, Y_{B_c}, C_c, D_c) \prec 0. \tag{27}$$

Here, $\widehat{L}(\cdot)$ is an affine function with respect to the decision variables $X \in \mathbf{S}_+^n$, $Z \in \mathbf{S}_+^n$, $Y_{A_c} \in \mathbf{R}^{n\times n}$, $Y_{B_c} \in \mathbf{R}^{n\times l_y}$, $C_c \in \mathbf{R}^{m_u\times n}$, and $D_c \in \mathbf{R}^{m_u\times l_y}$. In addition, it is also affine on the plant data M_γ (except for D_{12}). If this LMI is feasible, then the desired full-order H_∞ controller K can be reconstructed by

$$K = \left[\begin{array}{c|c} Z^{-1}Y_{A_c} & Z^{-1}Y_{B_c} \\ \hline C_c & D_c \end{array}\right]. \tag{28}$$

As noted, the BMI (23) can be linearized also by the known approaches in (Gahinet and Apkarian, 1994; Iwasaki and Skelton, 1994; Scherer et al., 1997; Masubuchi et al., 1998). However, in stark contrary to (28), the controller parametrization there involves state space matrices of the generalized plant and this is undesirable in several applications. One of the typical examples is the real μ synthesis. We will briefly discuss this issue in the sequel and show the usefulness of the present approach.

Let us consider the case where the state space matrices of G_γ is affected by the polytopic-type uncertainty as follows:

$$\begin{aligned} &\begin{bmatrix} A & B_1 & B_2 \\ C_{1,\gamma} & 0 & D_{12} \\ C_2 & 0 & 0 \end{bmatrix} \in \left\{ \sum_{l=1}^{L} \alpha_l \begin{bmatrix} A^{[l]} & B_1^{[l]} & B_2^{[l]} \\ C_{1,\gamma}^{[l]} & 0 & D_{12} \\ C_2^{[l]} & 0 & 0 \end{bmatrix} : \ \alpha \in \boldsymbol{\alpha} \right\}, \\ &\boldsymbol{\alpha} := \left\{ \alpha : \ \alpha \in \mathbf{R}^L, \ \sum_{l=1}^{L} \alpha_l = 1, \ \alpha_l \geq 0 \right\}. \end{aligned} \tag{29}$$

Here, $A^{[l]}$, $B^{[l]}$ $(l = 1, \cdots, L)$ and so on are known matrices. On the other hand, α is a time-invariant uncertain parameter whose only available information is $\alpha \in \boldsymbol{\alpha}$.

For ease of description, we denote by $G_\gamma(\alpha)$ the generalized plant for to the parameter $\alpha \in \boldsymbol{\alpha}$. Moreover, we define

$$M_\gamma^{[l]} := \{A^{[l]}, B_1^{[l]}, B_2^{[l]}, C_{1,\gamma}^{[l]}, D_{12}, C_2^{[l]}\} \ (l = 1, \cdots, L).$$

Then, our goal here is designing a full-order robust controller K satisfying

$$\|T(G_\gamma(\alpha), K)\|_\infty < 1 \quad \forall \alpha \in \boldsymbol{\alpha}. \tag{30}$$

Since the LMI (27) is affine with respect to the plant data M_γ other than D_{12}, and since the parametrization (28) does not depend on the plant data, such robust controller can be sought by solving the following LMI problem:

$$\widehat{L}(M_\gamma^{[l]}, X, Z, Y_{A_c}, Y_{B_c}, C_c, D_c) \prec 0 \ (l = 1, \cdots, L). \tag{31}$$

If this LMI is feasible, then the desired robust controller can be reconstructed via (28).

This approach is based on the well-known concept of quadratic stabilization (Bernussou et al., 1989), since we seek for a single Lyapunov matrix P of the form (24) that ensures the H_∞ performance over the whole uncertainty domain. Due to this restriction, the LMI approach (31) is surely conservative, but there is no other source of

conservatism. To this date, such effective and efficient quadratic-stability-based approach is only available for static state-feedback controller synthesis. In fact, for the present robust output-feedback H_∞ controller synthesis problem, we cannot apply the approaches in (Gahinet and Apkarian, 1994; Iwasaki and Skelton, 1994) from the outset since the elimination of the controller variables does not preserve the constraint that we have to generate a single (parameter-independent) robust controller. Similarly, the direct application of the approaches in (Scherer et al., 1997; Masubuchi et al., 1998) results in a controller that depends on α (and hence cannot be implemented). In the present approach, we have successfully circumvented these difficulties by exploiting the underlying assumptions on the generalized plant.

In order to reduce the conservatism of the above quadratic-stability-based approach, we further note that parameter dependent Lyapunov matrix $P(\alpha)$ of the form

$$P(\alpha) = \begin{bmatrix} X(\alpha) & Z \\ Z & Z \end{bmatrix} \tag{32}$$

can be employed. Here we need to remain Z to be constant since it is involved in the linearizing change of variables (26). If we resort to the parameter-dependent Lyapunov matrix (32), the resulting controller synthesis problem becomes a robust SDP for which powerful approaches are available at present. In particular, if we let $X(\alpha)$ to be linear on α and apply the idea of LMI dilation (Ebihara and Hagiwara, 2005), we can derive a tractable LMI problem which guarantees that we can yield better upper bound than the quadratic-stability-based approach. Details are omitted due to limited space.

3.3 Numerical Example

Let us consider again the mixed sensitivity problem discussed in Subsection 2.3. For the plant (11), we selected weightings as in (12) and constructed the generalized plant as in Fig. 3. For this generalized plant, we consider the problem (6) by concentrating on the full-order controllers. Obviously, this problem can be cast as the following SDP:

$$\inf_{X,Z,Y_{A_c},Y_{B_c},C_c,D_c} \quad \gamma \quad \text{subject to (27).}$$

By solving this SDP, we successfully designed an H_∞ optimal controller that achieves the optimal cost $\gamma_{\rm opt} = 0.35$. In order to verify this result, we also solved exactly the same problem by the standard approach in (Scherer et al., 1997; Masubuchi et al., 1998). Then, the optimal H_∞ performance again turns out to be 0.35 as expected.

We finally deal with the case where the parameters K_P and ζ of the plant (11) are uncertain but bounded as follows:

$$1.0 \le K_P \le 1.2, \quad 0.1 \le \zeta \le 0.2.$$

The corresponding generalized plant can be modeled as (29) by appropriately defining $M_\gamma^{[l]}$ $(l = 1, \cdots, 4)$. Under this setting, we aim at designing a robust H_∞ controller. More precisely, we want to solve

$$\gamma_{\rm rob} := \inf_K \quad \gamma \quad \text{subject to (30).}$$

To this end, we firstly solve the following SDP:

$$\inf_{X,Z,Y_{A_c},Y_{B_c},C_c,D_c} \quad \gamma \quad \text{subject to (31).}$$

Based on this quadratic-stability-based approach, an upper bound of $\gamma_{\rm rob}$ is computed as $\gamma_{\rm qd} = 0.8848$. Unfortunately, this result seems unsatisfactory due to the conservatism of the design strategy. To reduce the conservatism of the design, we next employed parameter-dependent Lyapunov matrix. Indeed, by following the procedure briefly sketched in the preceding subsection, we were able to design a suboptimal robust controller that achieves an upper bound $\gamma_{\rm pd} = 0.49$.

4. CONCLUSION

In this paper, we showed several structural conditions on the generalized plant under which the set of admissible static H_∞ controllers becomes convex. For the generalized plant satisfying these conditions, we further clarified that novel convexity results can be obtained even in the case of dynamical controller synthesis.

Although we have concentrated our attention on the continuous-time system synthesis, exactly the same convexity results can be obtained in the discrete-time setting under Assumption 1 (with obvious modification of the definition of stability). We finally note that Assumption 1 is of course only sufficient for the convexity of the set of admissible static H_∞ controllers. It is undoubtedly an important issue to investigate how to loosen this assumption.

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