Dual LMI Approach to Linear Positive System Analysis

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Abstract—This paper is concerned with the analysis of linear time-invariant continuous-time positive systems. Roughly speaking, a positive system is characterized by the property that its output is always nonnegative for any nonnegative input. Because of this strong property, there are remarkable, and very peculiar results that are valid only for positive systems. Among them, the existence of a diagonal Lyapunov matrix that characterizes stability is well known. Recently, this existence result has been extended to the linear matrix inequality (LMI) characterizing bounded realness of positive systems. The primal goal of this paper is to show that these results, as well as the celebrated Perron-Frobenius theorem, can be proved concisely via duality-based arguments, and in particular, along the line of the ‘rank-one separable property.’ This property has been the heart for deriving numerous LMI results for general linear system analysis. The implication of our new proofs is that, even for those very peculiar results for positive system analysis, the rank-one separable property again lies behind. In addition to these alternative proofs, we also derive several convex conditions for positive system analysis. These are effective particularly when we deal with robustness issues of positive systems with uncertain parameters. Furthermore, under a structural assumption on the uncertainty, we will show that we can solve these robustness analysis problems completely without any conservatism.

Keywords: positive system, LMI, duality, rank-one separable property.

I. INTRODUCTION

The analysis and synthesis of linear positive systems form an attractive research field in linear system theory. In general, a positive system is characterized by its peculiar input-to-output property, that is, its output and state, corresponding to any nonnegative input and nonnegative initial state, are always nonnegative. This property can be seen naturally in biology, network communications, economics and probabilistic systems. This wide range of application areas strongly motivates the study of positive systems [4], [9]. Even though practical systems in these fields are nonlinear in nature, it is known that linear positive system models are still valid in several applications, for example, in age-structured population models in demography [4].

Because of the strong nonnegative property, there are remarkable, and very peculiar results that are valid only for linear positive systems. Among them, the existence of a diagonal Lyapunov matrix characterizing stability is well

known [4], [9]. Recently, this existence result has been extended to the linear matrix inequality (LMI) characterizing bounded realness of positive systems [14]. The primal goal of this paper is to show that these results, as well as the celebrated Perron-Frobenius theorem [4], [9], can be proved concisely via duality-based arguments, and in particular, along the line of the *rank-one separable property*. This property has been the heart for deriving numerous LMI results for general linear system analysis. To this date, it has been shown that the Lyapunov inequality [6], the LMI in Kalman-Yakubovich-Popov (KYP) lemma [11], the LMI in generalized KYP lemma [8], and the generalized Lyapunov inequality [3] are all proved and derived along the line of the rank-one separable property. Our new proofs in this paper show that, even for those very peculiar results for positive system analysis, the rank-one separable property again lies behind.

It should be noted that there is a series of studies by Shorten et al. for positive system analysis using convex optimization [5], [10], [12], [13]. In particular, in [12], the diagonal stability of positive systems, or more precisely, the existence of diagonal Lyapunov matrices in the Lyapunov inequality for stable Metzler matrices, is proved by means of duality theory. Exactly the same line is pursued in [14]. These are indeed remarkable results. Nevertheless, their proofs involve Hadamar products among positive vectors and require several steps of peculiar algebraic manipulations. This might give an impression that, for positive system analysis, we need a completely different strategy from those for standard linear systems [6], [11], [8], [3]. Our primary contribution in this paper is to show that this is not the case. Indeed, we can interpret those results tailored for positive systems from the rank-one separable property, and this interpretation will be important certainly from a pedagogical point of view.

In addition to the alternative proofs, we also derive several convex conditions for stability and bounded realness analysis of positive systems. Since positive systems belong to a special class of general linear systems, our intuition is that it must be possible to somehow relax the standard *symmetricity* constraint on the Lyapunov matrix in the LMIs (contrary to restricting it to be diagonal). This intuition can indeed be justified, and roughly speaking, we will prove that the common positive definiteness constraint $P \succ 0$ can be relaxed to $P + P^T \succ 0$, which implies that P is not necessarily required to be symmetric. The extra freedom introduced by non-symmetry becomes effective particularly when we deal with robustness analysis problems against uncertain parameters in the system. Furthermore, under the *rank-one difference* assumption on the uncertainty, we will

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show that we can solve these problems completely without any conservatism. This result can be seen as an addition to the result in [10] for the existence of a common linear Lyapunov function.

We use the following notations in this paper. First, we denote by $\mathbb{S}_{++}^n$ ($\mathbb{S}_+^n$) the set of positive (semi)definite matrices of the size n . For symmetric matrix $X \in \mathbb{R}^{n \times n}$, we also write $X \succ 0$ ($X \succeq 0$) to denote $X \in \mathbb{S}_{++}^n$ ($X \in \mathbb{S}_+^n$). In addition, we denote by $\mathbb{D}_{++}^n$ the set of diagonal, and positive definite matrices of the size n . The notation $\lambda(A)$ stands for the set of the eigenvalues of $A \in \mathbb{R}^{n \times n}$. For given two matrices A and B of the same size, we write $A > B$ ($A \geq B$) if $A_{ij} > B_{ij}$ ($A_{ij} \geq B_{ij}$) holds for all (i, j) , where A_{ij} (B_{ij}) stands for the (i, j) -entry of A (B). We also define

$$\begin{aligned}\mathbb{R}_{++}^{n \times m} &:= \{A \in \mathbb{R}^{n \times m}, A > 0\}, \\ \mathbb{R}_+^{n \times m} &:= \{A \in \mathbb{R}^{n \times m}, A \geq 0\}.\end{aligned}$$

Finally, for given $A \in \mathbb{R}^{n \times n}$, we define by $\mathcal{D}(A) \in \mathbb{R}^n$ the vector composed of the diagonal entries, i.e., $\mathcal{D}(A) := [A_{11} \cdots A_{nn}]^T$.

II. REVIEW OF FUNDAMENTALS OF POSITIVE SYSTEMS

In this brief section, we gather very basic definitions and results for positive system analysis. See [4], [9] for complete treatment of positive systems.

Definition 1 (Externally Positive Linear System): [4] A linear system is said to be *externally positive* if its output is nonnegative for any nonnegative input if the initial state is zero.

Definition 2 (Positive Linear System): [4] A linear system is said to be *positive* if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.

Definition 3 (Metzler Martrix): [4] A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *Metzler* if its off-diagonal entries are all nonnegative, i.e., $A_{ij} \geq 0$ ($i \neq j$).

Theorem 1: [4] Let us consider the continuous-time LTI system described by

$$G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]. \quad (1)$$

Then, this system is positive if and only if A is Metzler, $B \geq 0$, $C \geq 0$, and $D \geq 0$.

By definition, a positive system is always an externally positive system, but the converse is not necessarily true [4]. In this paper we mainly discuss peculiar properties of positive linear systems, even though in several parts we refer to externally positive linear systems.

III. PRELIMINARY RESULTS

In this section, we introduce several preliminary results that are effective for positive system analysis. For conciseness, with a slight abuse of notation, we first make the following definition.

Definition 4: For given $H \in \mathbb{S}_+^n$, we define $\bar{h} \in \mathbb{R}_+^n$ by $\bar{h}_i = \sqrt{H_{ii}}$ ($i = 1, \dots, n$).

Under this definition, the following three lemmas hold.

Lemma 1: For given $H \in \mathbb{S}_+^n$, we have

$$(\bar{h}\bar{h}^T)_{ii} = H_{ii}, \quad (\bar{h}\bar{h}^T)_{ij} \geq H_{ij} \quad (i \neq j). \quad (2)$$

Proof: The first equality is obvious. On the other hand, since $H \succeq 0$, we have $H_{ii}H_{jj} \geq H_{ij}^2$ for $i \neq j$. It follows that $\sqrt{H_{ii}}\sqrt{H_{jj}} \geq H_{ij}$. Therefore, on the (i, j) entry of $\bar{h}\bar{h}^T - H$, we have

$$(\bar{h}\bar{h}^T)_{ij} - H_{ij} = \sqrt{H_{ii}}\sqrt{H_{jj}} - H_{ij} \geq 0.$$

This completes the proof. $\blacksquare$

Lemma 2: For given Metzler matrix $A \in \mathbb{R}^{n \times n}$ and $H \in \mathbb{S}_+^n$, we have $\mathcal{D}(\bar{h}\bar{h}^T A) \geq \mathcal{D}(HA)$.

Proof of Lemma 2: Since A is Metzler and hence $A_{ij} \geq 0$ ($i \neq j$), we see from Lemma 1 that

$$\begin{aligned}(\bar{h}\bar{h}^T A)_{ii} &= (\bar{h}\bar{h}^T)_{ii}A_{ii} + \sum_{j=1, j \neq i}^n (\bar{h}\bar{h}^T)_{ij}A_{ji} \\ &\geq H_{ii}A_{ii} + \sum_{j=1, j \neq i}^n H_{ij}A_{ji} \\ &= (HA)_{ii}.\end{aligned}$$

This completes the proof. $\blacksquare$

This lemma in particular implies that if there exists $H \in \mathbb{S}_+^n$ that satisfies $\mathcal{D}(HA) \geq 0$ for a given Metzler matrix $A \in \mathbb{R}^{n \times n}$, then, exactly the same property $\mathcal{D}(\bar{h}\bar{h}^T A) \geq 0$ holds with the *rank-one matrix* $\bar{h}\bar{h}^T$.

Lemma 3: For given Metzler matrix $A \in \mathbb{R}^{n \times n}$ and nonzero $h \in \mathbb{R}_+^n$, the following conditions are equivalent.

- (i) $\mathcal{D}(hh^T A) \geq 0$.
- (ii) $h^T A \geq 0$.

Proof of Lemma 3: Since (ii) $\Rightarrow$ (i) is obvious, we prove (i) $\Rightarrow$ (ii) by contradiction. To this end, suppose $(h^T A)_i < 0$. Then, since A is Metzler, we have $A_{ii} < 0$ and $h_i > 0$. Hence $h_i(h^T A)_i < 0$, which clearly contradicts (i). $\blacksquare$

IV. NEW PROOFS FOR DIAGONAL STABILITY OF METZLER MATRIX

The existing stability characterizations of Metzler matrices can be summarized as the following theorem [4], [9].

Theorem 2: [4], [9] Suppose $A \in \mathbb{R}^{n \times n}$ is Metzler. Then, the following conditions are equivalent.

- (i) The matrix A is Hurwitz stable.
- (ii) For any $h \in \mathbb{R}_+^n \setminus \{0\}$, the row vector $h^T A$ has at least one strictly negative entry.
- (iii) There exists $X \in \mathbb{D}_{++}^n$ such that $AX + XA^T \prec 0$.

The condition (iii) states that a Metzler matrix is Hurwitz stable if and only if it admits a diagonal Lyapunov matrix. This striking property is proved, for example in [4], via steady-state analysis of positive systems for positive and constant input. Our first contribution in this paper is to prove the condition (iii) along the spirit of the rank-one separable property. This property is frequently used in the proof of LMI-based results for linear system analysis [6], [11], [8], [3] and considered to be the central property that enables us to derive LMI characterizations. We prove that the rank-one separable property is again the heart for proving existing LMI results for positive system analysis. The particular difference is that, in contrast to the fact that the rank-one

matrix in [6], [11], [8], [3] is constructed from an eigenvector of the dual matrix variable (i.e., the Lagrange multiplier), the desired rank-one matrix for positive system analysis is constructed again from the dual variable but along the line of Definition 4. We emphasize that a duality-based proof for the LMIs of positive system analysis has been done by Shorten et al. [12] and this idea is followed by Tanaka and Langbort [14]. Nevertheless, an explicit connection to the rank-one separable property is missing, and this is indeed the main point we intend to clarify.

For the proof of Theorem 2, the implication (iii) $\Rightarrow$ (i) is obvious from the standard Lyapunov theory. Therefore we prove only (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii). Our contribution mainly lies in the latter proof.

Proof of Theorem 2:

(i) $\Rightarrow$ (ii) By contradiction, suppose there exists a nonzero $h \in \mathbb{R}_+^n$ such that $h^T A \geq 0$. With the Metzler matrix A , let us consider the autonomous internally positive system

$$\dot{x}(t) = Ax(t). \quad (3)$$

Then, if we define $V(x(t)) := h^T x(t)$, its derivative along the trajectory $\dot{x}(t) = Ax(t)$ for $x(0) \in \mathbb{R}_+^n$ satisfies

$$\frac{dV(x(t))}{dt} = h^T \dot{x}(t) = h^T Ax(t) \geq 0 \quad (\forall t \geq 0).$$

Here, we used the fact that $x(t) \geq 0$ ($\forall t \geq 0$). It follows that the trajectory starting from $x(0) = h$, for which $V(x(0)) > 0$, fails to converge to the origin because otherwise $V(x(t))$ should tend to 0 as $t \rightarrow \infty$. This proves that the matrix A is not Hurwitz, which contradicts (i).

(ii) $\Rightarrow$ (iii) Again, by contradiction, suppose (iii) does not hold for any $X \in \mathbb{D}_{++}^n$. Then, from the separating hyperplane theorem [2], there exists a nonzero $H \in \mathbb{S}_{++}^n$ such that

$$\text{trace}((HA + A^T H)X) \geq 0 \quad \forall X \in \mathbb{D}_{++}^n.$$

This clearly shows $\mathcal{D}(HA) \geq 0$. It follows from Lemma 2 that $\mathcal{D}(\bar{h}\bar{h}^T A) \geq 0$, where $\bar{h} \in \mathbb{R}_+^n \setminus \{0\}$ is defined from Definition 4. From Lemma 3, this implies $\bar{h}^T A \geq 0$, which contradicts (ii). ■

In the above proof for (ii) $\Rightarrow$ (iii), the most important step is to show that if $\mathcal{D}(HA) \geq 0$, then exactly the same nonnegative condition $\mathcal{D}(\bar{h}\bar{h}^T A) \geq 0$ is satisfied by the rank-one matrix $\bar{h}\bar{h}^T$. Once this is established, the rest of the proof is fairly standard. In this way, the proof for the existence of diagonal Lyapunov matrices is completed in a streamlined fashion, based on duality-based arguments in conjunction with the rank-one separable property.

It should be noted that, by taking the dual of the condition (ii), we can also derive the following well-known result for characterizing the Hurwitz stability of Metzler matrices [10].

Theorem 3: Suppose $A \in \mathbb{R}^{n \times n}$ is Metzler. Then, the following conditions are equivalent to (i) of Theorem 2.

- (iv) There exists $g_R \in \mathbb{R}_{++}^n$ such that $Ag_R < 0$.
- (v) There exists $g_L \in \mathbb{R}_{++}^n$ such that $g_L^T A < 0$.

In the following, we will give a complete proof for this theorem again based on duality. The equivalence between (iv) and (v) is obvious if we consider A^T , which is Metzler

and Hurwitz if and only if A is. Hence, we only show the equivalence between (ii) in Theorem 2 and (iv) in Theorem 3.

Proof of Theorem 3: (iv) $\Rightarrow$ (ii) Suppose (iv) holds. Then, for every nonzero $h \in \mathbb{R}_+^n$, we have $h^T Ag_R < 0$. Since $g_R > 0$, this implies $h^T A$ has at least one strictly negative entry and therefore (ii) holds.

(ii) $\Rightarrow$ (iv) By contradiction, suppose (iv) does not hold for any $g_R > 0$. Then, from the strong alternatives for linear inequalities [2], we see that there exists nonzero $h \in \mathbb{R}^n$ such that $h \geq 0$, $h^T A \geq 0$. This clearly contradicts (ii). ■

We finally note that the condition (v) indicates the well-known fact that an autonomous internally positive system (3) is stable if and only if it admits a linear Lyapunov function of the form $V(x(t)) = g_L^T x(t)$ ($g_L > 0$) that certifies stability.

V. NEW PROOFS FOR PERRON-FROBENIUS THEOREM

The next theorem is widely known as the Perron-Frobenius Theorem.

Theorem 4 (Perron-Frobenius Theorem): [4], [9] Suppose $A \in \mathbb{R}^{n \times n}$ is nonnegative, i.e., $A \geq 0$. Then, A has a nonnegative eigenvalue α such that $\alpha = \max_{\lambda \in \lambda(A)} |\lambda|$.

This theorem is undoubtedly the central result in positive system analysis. It has a vast range of application areas such as biology, sociology and stochastic system analysis. This theorem is proved, for example in [9], by means of the Collatz-Wielandt function. Our contribution in this section is to show that this celebrated theorem can be proved concisely by means of the rank-one separable property.

Proof of Theorem 4: Let us denote by ρ the spectral radius of A , i.e., $\rho := \max_{\lambda \in \lambda(A)} |\lambda|$. Then, it is a direct consequence from the discrete-time Lyapunov inequality that there exists no $P \succ 0$ such that $\rho^2 P - APA^T \succ 0$. From duality, or more specifically, from strong alternatives for generalized inequalities [1], [2], this can be restated as the assertion that there exists nonzero H such that

$$H \succeq 0, \quad A^T H A - \rho^2 H \succeq 0.$$

The latter inequality implies that $\mathcal{D}(A^T H A) \geq \mathcal{D}(\rho^2 H)$. Here let us define $\bar{h} \in \mathbb{R}_+^n \setminus \{0\}$ from H as in Definition 4 and recall that $\bar{h}\bar{h}^T \geq H$ and $\mathcal{D}(\bar{h}\bar{h}^T) = \mathcal{D}(H)$. Since $A \geq 0$, we have $A^T \bar{h}\bar{h}^T A \geq A^T H A$. Hence, we see that $\mathcal{D}(A^T \bar{h}\bar{h}^T A) \geq \mathcal{D}(A^T H A) \geq \mathcal{D}(\rho^2 H) = \mathcal{D}(\rho^2 \bar{h}\bar{h}^T)$ holds. Furthermore, since $\bar{h}^T A \geq 0$ and $\rho \bar{h}^T \geq 0$, we have $\bar{h}^T A \geq \rho \bar{h}^T$ or equivalently, $\bar{h}^T (A - \rho I) \geq 0$. From (ii) of Theorem 2, this indicates that $A - \rho I$ is not Hurwitz stable. Since ρ is the spectral radius of A , this occurs only by the existence of an eigenvalue of A at ρ . ■

It should be noted that a more general form of the Perron-Frobenius Theorem asserts that the eigenvector g corresponding to the eigenvalue $\alpha = \rho$ satisfies $g \geq 0$ [†]. The existence of such a nonnegative eigenvector can be shown again via duality-based arguments. In the following, we will give a brief sketch of the proof.

[†]Under the assumption that A is irreducible, the Perron-Frobenius Theorem ensures $g > 0$ that is stronger than $g \geq 0$. See [4], [9] for details.

By contradiction, suppose there exists no $g \in \mathbb{R}_+^n \setminus \{0\}$ such that $(A - \rho I)g = 0$. Then, from the strong alternative for linear inequality [2], there exists $h \neq 0$ such that $h^T(A - \rho I) > 0$. Here, if $h \leq 0$, then we have $-h^T(A - \rho I) < 0$, and by perturbing h , we see that there exists $\tilde{h} > 0$ such that $\tilde{h}^T(A - \rho I) < 0$. From Theorem 3, this contradicts the fact that $A - \rho I$ is not Hurwitz. Therefore it suffices to consider the case where h has at least one strictly positive entry.

With this in mind, let us denote by $h_+ \in \mathbb{R}_+^n \setminus \{0\}$ the projection of h onto $\mathbb{R}_+^n$. We also define h_- by $h_- := h - h_+ \leq 0$. Since $h^T(A - \rho I) > 0$, there exists $\varepsilon > 0$ such that $v := h^T(A - (\rho + \varepsilon)I) > 0$, and this implies $v_+ := h_+^T(A - (\rho + \varepsilon)I) \geq 0$ holds as well. This is because if v_+ has a negative entry, i.e., $v_{+,j} = (h_+^T A)_j - (\rho + \varepsilon)h_{+,j} < 0$, then it is obvious that $h_{+,j} > 0$ and hence $h_{-,j} = 0$. It follows that $v_- := h_-^T(A - (\rho + \varepsilon)I)$ satisfies $v_{-,j} = (h_-^T A)_j \leq 0$. This clearly contradicts the fact that $v_j = v_{+,j} + v_{-,j} > 0$. To summarize, we have established the existence of $h_+ \neq 0$ such that $h_+ \geq 0$ and $h_+^T(A - (\rho + \varepsilon)I) \geq 0$. From (ii) in Theorem 2, this contradicts the fact that $A - (\rho + \varepsilon)I$ is Hurwitz stable. This completes the proof.

VI. NEW STABILITY CONDITION FOR METZLER MATRIX

In this subsection, we derive a new LMI condition for Hurwitz stability of Metzler matrices. To this end, let us recall the following lemma.

Lemma 4: [4], [9] Suppose $A \in \mathbf{R}^{n \times n}$ is Metzler. Then, A has a real eigenvalue α such that $\alpha = \max_{\lambda \in \lambda(A)} \text{Re} \lambda$.

This lemma is a direct consequence of Theorem 4, i.e., the Perron-Frobenius theorem. Indeed, since A is Metzler, there exists $\beta \geq 0$ such that $A + \beta I \geq 0$. Then, from Theorem 4, the matrix $A + \beta I$ has a nonnegative eigenvalue s such that $s = \max_{\lambda \in \lambda(A + \beta I)} |\lambda|$ and therefore $s = \max_{\lambda \in \lambda(A + \beta I)} \text{Re} \lambda$. This clearly shows the existence of α in Lemma 4 that is precisely given by $\alpha = s - \beta$.

From this lemma, we see that a Metzler matrix is Hurwitz stable if and only if all its real eigenvalues are negative. On the other hand, regarding the location of real eigenvalues for general (not necessarily Metzler) matrices, we can derive the following lemma.

Lemma 5: Suppose $A \in \mathbf{R}^{n \times n}$ has at least one real eigenvalue. Then, all real eigenvalues of A are negative if and only if there exists $M \in \mathbf{R}^{n \times n}$ such that

$$M + M^T \succ 0, \quad AM + M^T A^T \prec 0. \quad (4)$$

Proof: (Sufficiency) Suppose $(\lambda, \xi) \in \mathbb{R} \times \mathbb{R}^n$ is an eigenvalue-eigenvector pair of A such that $\xi^T A = \lambda \xi^T$. Then, if the latter inequality in (4) holds, we have

$$\xi^T (AM + M^T A^T) \xi = \lambda \xi^T (M + M^T) \xi < 0.$$

From the former inequality in (4), we have $\xi^T (M + M^T) \xi > 0$. It follows that $\lambda < 0$.

(Necessity) By contradiction, suppose (4) does not hold for any $M \in \mathbf{R}^{n \times n}$. Then, there exists $H, G \in \mathbb{S}_+^n$, not simultaneously zero, such that

$$\text{trace} M(HA - G) \geq 0 \quad \forall M \in \mathbf{R}^{n \times n}.$$

Hence, we have $HA - G = 0$. This implies that $HA \succeq 0$ with nonzero $H \succeq 0$. If we denote by $H_1 H_1^T$ the full-rank factorization of H , we have $H_1 H_1^T A \succeq 0$. Therefore there exists $W \in \mathbb{S}_+^q$ such that $H_1^T A = W H_1^T$ where $q := \text{rank}(H_1)$. If we denote the spectral factorization of W by $U \Lambda U^T$, we arrive at $U^T H_1^T A = \Lambda U^T H_1^T$. This shows that A has at least one nonnegative real eigenvalue. ■

From Lemmas 4 and 5, we arrive at the next theorem that provides a new LMI condition for the stability of Metzler matrices.

Theorem 5: Suppose $A \in \mathbf{R}^{n \times n}$ is Metzler. Then, the following condition is equivalent to (i) of Theorem 2.

(vi) There exists $M \in \mathbf{R}^{n \times n}$ such that (4) holds.

From the standard Lyapunov theory and Theorem 2, we see that the condition (vi) in Theorem 5 is still necessary and sufficient even if we restrict M in (4) to be symmetric or even to be diagonal. The extra freedom introduced by relaxing M to be a non-symmetric matrix becomes effective, for example, when we deal with robust stability analysis problem against parametric uncertainties in the matrix A . See Section VIII for more concrete discussion.

VII. DIAGONAL KYP LEMMA FOR POSITIVE SYSTEMS

A. New Proofs for Known Results

The following result, which ensures the existence of diagonal Lyapunov matrix in the LMI characterizing bounded-realness of positive systems, was firstly shown by Tanaka and Langbort [14].

Theorem 6: [14] Let us consider the positive system described by (1). Then, the following conditions are equivalent.

(i) The matrix A is Hurwitz and $\|G\|_\infty < 1$.

(ii) There exists $X \in \mathbb{D}_{++}^n$ such that

$$\begin{bmatrix} AX + XA^T + BB^T & XC^T + BD^T \\ CX + DB^T & DD^T - I \end{bmatrix} \prec 0. \quad (5)$$

The proof for this theorem given in [14] involves Hadamard product among positive vectors and requires several steps of algebraic manipulations. Here, we again provide a streamlined proof by relying on the rank-one separable property. Since the implication (ii) $\Rightarrow$ (i) is obvious from the KYP lemma, the issue is how to prove (i) $\Rightarrow$ (ii).

Proof of Theorem 6: (i) $\Rightarrow$ (ii) By contradiction, suppose (ii) does not hold for any $X \in \mathbb{D}_{++}^n$. Then, from the separating hyperplane theorem, there exists nonzero $H \in \mathbb{S}_+^{n+l}$ such that

$$\text{trace} \left(H \begin{bmatrix} AX + XA^T + BB^T & XC^T + BD^T \\ CX + DB^T & DD^T - I \end{bmatrix} \right) \geq 0 \quad (6)$$

$$\forall X \in \mathbb{D}_{++}^n.$$

or equivalently,

$$2\text{trace}((H_{11}A + H_{12}C)X) + \text{trace} \left(H \begin{bmatrix} BB^T & BD^T \\ DB^T & DD^T - I \end{bmatrix} \right) \geq 0 \quad \forall X \in \mathbb{D}_{++}^n,$$

$$H =: \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}, \quad H_{11} \in \mathbb{S}_+^n, \quad H_{22} \in \mathbb{S}_+^l.$$

This indicates that

- $\mathcal{D}(H_{11}A + H_{12}C) \geq 0$, and
- $\text{trace} \left(H \begin{bmatrix} BB^T & BD^T \\ DB^T & DD^T - I \end{bmatrix} \right) \geq 0$.

We note that the first condition is equivalent to

- $\mathcal{D} \left(H \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \right) \geq 0$.

Then, since both

$$\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \begin{bmatrix} BB^T & BD^T \\ DB^T & DD^T - I \end{bmatrix}$$

are Metzler, we see from Lemma 2 that $\bar{h} \in \mathbb{R}_+^{n+l} \setminus \{0\}$ defined from H as in Definition 4 satisfies

- $\mathcal{D} \left(\bar{h} \bar{h}^T \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \right) \geq 0$, and
- $\text{trace} \left(\bar{h} \bar{h}^T \begin{bmatrix} BB^T & BD^T \\ DB^T & DD^T - I \end{bmatrix} \right) \geq 0$.

If we further define $\bar{h} = [\bar{h}_1^T \bar{h}_2^T]^T$ where $\bar{h}_1 \in \mathbb{R}_+^n$ and $\bar{h}_2 \in \mathbb{R}_+^l$, it follows that

- (a) $\mathcal{D}(\bar{h}_1(\bar{h}_1^T A + \bar{h}_2^T C)) \geq 0$, and
- (b) $(\bar{h}_1^T B + \bar{h}_2^T D)(\bar{h}_1^T B + \bar{h}_2^T D)^T \geq \bar{h}_2^T \bar{h}_2$.

The rest of the proof is exactly the same as in [14]. Nevertheless, here we will repeat the argument in [14] since we use this in the next subsection to derive novel results for characterizing bounded realness of positive systems.

If $\bar{h}_2 = 0$, then $\bar{h}_1 \neq 0$ and $\mathcal{D}(\bar{h}_1 \bar{h}_1^T A) \geq 0$ from (a). From Lemma 3, this implies $\bar{h}_1^T A \geq 0$ and from Theorem 2, this contradicts the assertion in (i) that A is Hurwitz. Therefore, we assume $\bar{h}_2 \neq 0$, and hence $\bar{h}_1^T B + \bar{h}_2^T D \neq 0$ from (b). Then, we note that

$$\bar{h}_2^T = (\bar{h}_1^T B + \bar{h}_2^T D) \Delta \quad (7)$$

holds for

$$\Delta = (\bar{h}_1^T B + \bar{h}_2^T D)^T h_2^T / (\bar{h}_1^T B + \bar{h}_2^T D)(\bar{h}_1^T B + \bar{h}_2^T D)^T,$$

which obviously satisfies $\Delta \geq 0$, and (b) implies $\|\Delta\| \leq 1$ as well. Since $\bar{h}_2^T = \bar{h}_1^T B \Delta (I - D \Delta)^{-1}$ holds from (7), we have $\mathcal{D}(\bar{h}_1 \bar{h}_1^T \tilde{A}) \geq 0$ from (a) where $\tilde{A} := A + B \Delta (I - D \Delta)^{-1} C$. Moreover, since we can confirm that $(I - D \Delta)^{-1} \geq 0$ holds from its Taylor series expansion, it is obvious that $\tilde{A}$ is Metzler. Therefore, from $\mathcal{D}(\bar{h}_1 \bar{h}_1^T \tilde{A}) \geq 0$, we see that $\tilde{A}$ is unstable. To summarize, we have build a norm bounded Δ such that $\|\Delta\| \leq 1$ that renders the matrix $\tilde{A} = A + B \Delta (I - D \Delta)^{-1} C$ unstable. This contradicts $\|G\|_\infty < 1$. ■

B. New Conditions for Bounded Realness

Now we are ready to show new conditions for characterizing bounded realness of positive systems.

Theorem 7: Under the same assumptions and notations of Theorem 6, the following two conditions are equivalent to the condition (i) in Theorem 6.

- (iii) The matrix A is Hurwitz and $\|G(0)\| = \|-CA^{-1}B + D\| < 1$.
- (iv) There exists $M \in \mathbf{R}^{n \times n}$ such that

$$\begin{aligned} M + M^T &\succ 0, \\ \begin{bmatrix} AM + M^T A^T + BB^T & M^T C^T + BD^T \\ CM + DB^T & DD^T - I \end{bmatrix} &\prec 0. \end{aligned} \quad (8)$$

Proof of Theorem 7: We will show the equivalence to the conditions to (i) and (ii) in Theorem 6. To prove the validity of (iii), we first note that (i) $\Rightarrow$ (iii) is obvious. Therefore it suffices to prove (iii) $\Rightarrow$ (ii) ($\Leftrightarrow$ (i)).

(iii) $\Rightarrow$ (ii) By contradiction, suppose (ii) does not hold. Then, from the proof of Theorem 6, we arrive at the conclusion that

- (a) the matrix A is not Hurwitz stable, or
- (b) there exists $\Delta \in \mathbb{R}_+^{m \times l}$ with $\|\Delta\| \leq 1$ such that $\tilde{A} = A + B \Delta (I - D \Delta)^{-1} C$ is Metzler and unstable.

Since (a) clearly contradicts (iii), it suffices to consider the case (b). Therefore, let us assume that (b) holds. Then, we first note that $\tilde{A}(\beta) := A + B \beta \Delta (I - D \beta \Delta)^{-1} C$ is Metzler for any $\beta \in [0, 1]$. In particular, it is obvious that $\tilde{A}(0) = A$ is Hurwitz stable, and $\tilde{A}(1)$ is unstable. Therefore, from Lemma 4 and the continuity of the eigenvalues of $\tilde{A}(\beta)$, we see that there exists $\beta^* \in [0, 1]$ such that $\det(\tilde{A}(\beta^*)) = 0$. It follows that

$$\det \left(\begin{bmatrix} A & B \beta \Delta \\ -C & I - D \beta \Delta \end{bmatrix} \right) = 0.$$

This implies $\det(I - (-CA^{-1}B + D)\beta \Delta) = 0$. Since $\|\beta \Delta\| \leq 1$, the above condition holds only if $\|-CA^{-1}B + D\| \geq 1$. This clearly contradicts (iii).

On the other hand, for the validity of (iv), we note that (ii) $\Rightarrow$ (iv) is obvious. Therefore we will prove (iv) $\Rightarrow$ (iii) ($\Leftrightarrow$ (ii)).

(iv) $\Rightarrow$ (iii) Suppose (iv) holds. Then, we have

$$M + M^T \succ 0, \quad AM + M^T A^T \prec 0.$$

Therefore, from Theorem 5, we see that A is Hurwitz stable. Moreover, the latter inequality in (8) can be rewritten equivalently as

$$\begin{bmatrix} BB^T & BD^T \\ DB^T & DD^T - I \end{bmatrix} + \text{He} \left\{ \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} M & 0 \end{bmatrix} \right\} \prec 0.$$

It follows that

$$\begin{bmatrix} A^{-T} C^T \\ -I \end{bmatrix}^T \left(\begin{bmatrix} BB^T & BD^T \\ DB^T & DD^T - I \end{bmatrix} + \text{He} \left\{ \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} M & 0 \end{bmatrix} \right\} \right) \begin{bmatrix} A^{-T} C^T \\ -I \end{bmatrix} = (-CA^{-1}B + D)(-CA^{-1}B + D)^T - I \prec 0.$$

It clearly shows that $\|-CA^{-1}B + D\| < 1$. ■

The following corollary is a direct consequence of Theorem 7.

Corollary 1: Under the notation of Theorem 6, we have $\|G\|_\infty = \|-CA^{-1}B + D\| = \|G(0)\|$ (9) if A is Hurwitz.

To summarize this subsection, we first prove that the H_∞ norm of a continuous-time positive system coincides with the matrix norm of its steady-state gain. We note that this result is known for discrete-time positive systems [7]. Even for the continuous-time positive systems, the result is almost obvious in SISO case. This is because, for the nonnegative input $1 + \sin \omega t$, its steady state output is given of the form $|G(0)| + |G(j\omega)| \sin(\omega t + \theta_\omega)$, and this should be nonnegative

for every $\omega > 0$. Indeed, by pursuing this input-to-output property in the time-domain, we can prove that (9) holds even for MIMO *externally* positive systems. On the other hand, we also clarified in Theorem 7 that, when characterizing bounded realness of positive systems, we can loosen the common symmetric structure of the variable in KYP-LMI to be non-symmetric. As discussed in Section VIII, the extra freedom will be effective when we deal with robust bounded realness analysis of positive systems.

VIII. ROBUST STABILITY AND ROBUST BOUNDED REALNESS ANALYSIS

In this section, we consider the case where the coefficient matrices of the positive system (1) are affected by uncertain parameter $\alpha \in \alpha \in \mathbb{R}^N$ as in

$$G_\alpha(s) := \left[\begin{array}{c|c} A(\alpha) & B(\alpha) \\ \hline C(\alpha) & D(\alpha) \end{array} \right],$$

$$\left[\begin{array}{cc} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{array} \right] = \sum_{i=1}^N \alpha_i \left[\begin{array}{cc} A_i & B_i \\ C_i & D_i \end{array} \right], \quad (10)$$

$$\alpha = \left\{ \alpha \in \mathbb{R}_+^N : \sum_{i=1}^N \alpha_i = 1 \right\}.$$

Here, $A_i \in \mathbb{R}^{n \times n}$ ($i = 1, \dots, N$), etc., are given matrices, satisfying that $A_i \in \mathbb{R}^{n \times n}$ is Metzler, and $B_i \in \mathbb{R}_+^{n \times m}$, $C_i \in \mathbb{R}_+^{l \times n}$, and $D_i \in \mathbb{R}_+^{l \times m}$. This is nothing but assuming that the system G_α is positive irrespective of $\alpha \in \alpha$. Our goal in this subsection is to derive numerically tractable and less conservative conditions for the robust stability and robust bounded realness analysis of G_α . For this aim, the next two results readily follow from Theorems 5 and 7.

Theorem 8: The positive system G_α defined by (10) is stable for all $\alpha \in \alpha$ if there exists $M \in \mathbb{R}^{n \times n}$ such that $M + M^T \succ 0$, $A_i M + M^T A_i^T \prec 0$ ($i = 1, \dots, N$).

Theorem 9: The positive system G_α defined by (10) satisfies $\|G_\alpha\|_\infty < 1$ for all $\alpha \in \alpha$ if there exists $M \in \mathbb{R}^{n \times n}$ such that $M + M^T \succ 0$,

$$\left[\begin{array}{cc} A_i M + M^T A_i^T + B_i B_i^T & M^T C_i^T + B_i D_i^T \\ C_i M + D_i B^T & D_i D_i^T - I \end{array} \right] \prec 0.$$

The validity of these theorems can be confirmed via simple convexity arguments. Obviously, the LMI conditions in Theorems 8 and 9 are less conservative (or more precisely, no worse) than those LMIs with symmetric, or diagonal Lyapunov matrices. Indeed, in our tested numerical examples, the conditions in Theorems 8 and 9 indeed work effectively to obtain less conservative results.

Even though Theorems 8 and 9 can be applied to the uncertain system (10) without any additional assumptions, the conditions are still conservative in general. On the other hand, under the additional *rank-one difference* assumption, we can provide definite answer for the robust stability analysis problem we posed. The key result is stated in the following.

Theorem 10: For given $A_0 \in \mathbb{R}^{n \times n}$, $\mathcal{E} \in \mathbb{R}_+^{n \times 1}$, $\mathcal{F}_i \in \mathbb{R}_+^{1 \times n}$ ($i = 1, \dots, N$) with A_0 Metzler and Hurwitz stable, let us define $\mathcal{A}_i := A_0 + \mathcal{E} \mathcal{F}_i$ ($i = 1, \dots, N$). Then, the following conditions are equivalent.

- (i) The matrices $\mathcal{A}_i$ ($i = 1, \dots, N$) are all Hurwitz stable.
- (ii) There exists $g \in \mathbb{R}_{++}^n$, which is independent of i , such that $\mathcal{A}_i g < 0$ ($i = 1, \dots, N$).
- (iii) $\mathcal{A}(\alpha) := \sum_{i=1}^N \alpha_i \mathcal{A}_i$ is Hurwitz stable for all $\alpha \in \alpha$.

Proof: Since the implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are obvious, we will prove (i) $\Rightarrow$ (ii) via contradiction. To this end, suppose (ii) does not hold. Then, from duality, there exist $h_i \in \mathbb{R}_+^n$ ($i = 1, \dots, N$), not simultaneously zero, such that

$$\sum_{i=1}^N h_i^T (\mathcal{A}_0 + \mathcal{E} \mathcal{F}_i) \geq 0. \quad (11)$$

We note that $h_i^T \mathcal{E} \neq 0$ for at least one index i since $\mathcal{A}_0$ is Metzler and Hurwitz stable. Furthermore, since $\mathcal{A}_0^{-1} \mathcal{E} \leq 0$, we see that (11) implies $\sum_{i=1}^N h_i^T \mathcal{E} (1 + \mathcal{F}_i \mathcal{A}_0^{-1} \mathcal{E}) \leq 0$. Since $h_i^T \mathcal{E} \geq 0$ ($i = 1, \dots, N$), and $h_i^T \mathcal{E} \neq 0$ for at least one index i as noted, it follows that $1 + \mathcal{F}_{i^*} \mathcal{A}_0^{-1} \mathcal{E} \leq 0$ holds for at least one index i^* . This implies that $\mathcal{A}_0 + \mathcal{E} \mathcal{F}_{i^*}$ is not Hurwitz stable, since for $h := -(\mathcal{F}_{i^*} \mathcal{A}_0^{-1})^T \geq 0$ we have $h^T (\mathcal{A}_0 + \mathcal{E} \mathcal{F}_{i^*}) = -(1 + \mathcal{F}_{i^*} \mathcal{A}_0^{-1} \mathcal{E}) \mathcal{F}_{i^*} \geq 0$. ■

Remark 1: This theorem is closely related to Theorem 4.1 of [10], where the existence of common linear Lyapunov function for two Metzler and Hurwitz stable matrices with rank-one difference is discussed. Between these two theorems, there seems no clear inclusion relationship. It should be noted that, under the assumption of Theorem 10, we *cannot* in general ensure the existence of $h > 0$ such that $h^T \mathcal{A}_i < 0$ ($i = 1, \dots, N$). Namely, under the assumption that the matrix $\mathcal{F} \in \mathbb{R}_+^{1 \times n}$ depends on the index i in Theorem 10, we cannot ensure the existence of common linear Lyapunov function of the form $V(x) = h^T x$ for the set of autonomous positive systems $\dot{x}(t) = \mathcal{A}_i x(t)$ ($i = 1, \dots, N$).

A direct consequence of this theorem is stated in the following corollary.

Corollary 2: Suppose A_i ($i = 1, \dots, N$) in (10) is given by $A_i = A_0 + E F_i$ where $A_0 \in \mathbb{R}^{n \times n}$, $E \in \mathbb{R}_+^{n \times 1}$, $F_i \in \mathbb{R}_+^{1 \times n}$ ($i = 1, \dots, N$) with A_0 Metzler and Hurwitz stable. Then, the system G_α is stable for all $\alpha \in \alpha$ if and only if A_i ($i = 1, \dots, N$) are all stable.

Thus, under the additional rank-one difference assumption, we can solve the robust stability analysis problem completely without any conservatism.

IX. CONCLUSION

In this paper, we clarified that several remarkable and peculiar results for positive system analysis can be proved via duality-based arguments, along the line of the rank-one separable property. The alternative proofs given in this paper might not be innovative, but still, we believe that they are useful for understanding those peculiar results from a standard, solid, and unified viewpoint. We also derived new LMI conditions for robust stability and robust bounded realness analysis of uncertain positive systems. In addition, under the rank-one difference assumption, we proved that the robust stability analysis problem can be solved completely, which has never been attained for general linear systems.

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