Global asymptotic stabilization of systems satisfying two different sector conditions

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Abstract

Global asymptotic stabilization for a class of nonlinear systems is addressed. The dynamics of these systems are composed of a linear part to which is added some nonlinearities which satisfy two different sector bound conditions depending on whether the state is near of far from the origin. The proposed approach is based on the uniting of control Lyapunov functions. In this framework, the stabilization problem may be recast as an LMI optimization problem for which powerful semidefinite programming softwares exist. This is illustrated by means of three numerical examples.

1. Introduction

There is an extensive literature on the design of nonlinear stabilizers providing numerous techniques which apply on specific classes of nonlinear systems. The class of systems under interest in this paper is the one described by nonlinear functions satisfying sector bound conditions. This class of nonlinearity includes many different memoryless functions (see e.g., [10, Chapter 6] for an introduction on this topic) such as saturations (see e.g. [18, 9, 8] for design techniques of control system with such nonlinearities). To oppose to what has been done in these papers, two different sector conditions are considered to characterize the nonlinear functions: one sector condition when the state is near the equilibrium and one other sector condition when the state is far from the equilibrium. This distinction between small and large values of the distance from the state to the equilibrium allows us to better describe the nonlinear system. Moreover we remark that encompassing both sector conditions into one global sector condition may lead to a too conservative synthesis problem which may not have a solution (see the example of Section 5.1 below).

This motivates us to separately consider the local sector condition and the non-local one. In our approach, we design successively:

1. a local stabilizer with a basin of attraction containing a compact set and
2. a non-local controller such that the previous compact set is globally attractive.

In a second step, in order to design a continuous global stabilizer, the local and non-local controllers are merged into one unique and global controller. Different techniques exist to unit a couple of different feedback laws. For instance, provided that the use of discontinuous

controllers is allowed, hybrid controllers may be employed to unit them (see [13, 14, 15, 20]). In the present paper we apply the technique introduced in [2] where a continuous solution to the uniting problem is given through the construction of a uniting control Lyapunov function[1].

More precisely, in [2] some sufficient conditions are given to provide a global stabilizer from a local and a non-local control Lyapunov function[2]. In the following, we rewrite these sufficient conditions in terms of linear matrix inequalities (LMIs) which, if solved, allow to design a global stabilizer for the control of systems satisfying two different sector conditions.

To be more precise, consider the system defined by its state-space equation:

$$\dot{x} \;=\; A\,x \,+\, B\,u \,+\, G\,\phi(x) \tag{1}$$

where the state vector x is in $\mathbb{R}^n$. (A, B, G) are matrices respectively in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$, $\mathbb{R}^{n\times p}$. Moreover u in $\mathbb{R}^m$ is the control input and $\phi(x) : \mathbb{R}^n \to \mathbb{R}^p$ is a nonlinear locally Lispchitz function such that $\phi(0) = 0$.

One way to design a global stabilizer for system (1) is to use circle and Popov criteria (see [4]) under the assumption that the nonlinear function ϕ satisfies some sector bound conditions of the following type[3]:

$$(\phi(x) - Mx)'(\phi(x) - Nx) \leq 0 \;, \forall\, x \;\in\; \mathbb{R}^n \;, \tag{2}$$

where M and N are two given matrices in $\mathbb{R}^{p\times n}$. Following[4] [6, 12], a constructive LMI condition allowing to design a state feedback control law solving the stabilizing problem may be exhibited.

The aim of this paper is to study the case in which the function ϕ satisfies two different sector conditions depending on the norm of x. The idea of the design is then to apply techniques inspired by [2] to unite local and non-local controllers and to provide a global stabilizer.

Assumptions on the nonlinear function ϕ introduced in (1) can be given as follows:

Assumption 1. Local sector condition. *There exist a positive real number v_0, two matrices (M_0, N_0) in $\mathbb{R}^{p\times n} \times \mathbb{R}^{p\times n}$ such that, for all $|x| \leq v_0$, we have:*

$$(\phi(x) - M_0\,x)'\,(\phi(x) - N_0\,x) \;\leq\; 0 \;. \tag{3}$$

Assumption 2. Non-local sector condition. *There exist a positive real number $v_\infty < v_0$, two matrices (M_∞, N_∞) in $\mathbb{R}^{p\times n} \times \mathbb{R}^{p\times n}$ and two vectors (R_∞, Q_∞) in $\mathbb{R}^p \times \mathbb{R}^p$ such that, for all $|x| \geq v_\infty$, we have:*

$$(\phi(x) - M_\infty\,x - R_\infty)'\,(\phi(x) - N_\infty\,x - Q_\infty) \;\leq\; 0 \;. \tag{4}$$

[1]see [5] for a definition of CLF.

[2]see [2] for definitions of local and non-local CLFs.

[3]Note that other classes of sector conditions are possible. In particular we may consider the generalized sector conditions written as

$$(\phi(x) - M\,x)'D\,(\phi(x) - N\,x) \;\leq\; 0 \;,$$

where D is any given diagonal positive definite matrix (as in [7, 19]). Despite the fact that considering such generalized sector conditions is possible, we restrict our attention to sector bound condition as (2) to ease the exposition of our results.

[4]Note that in [6] is addressed a more involved problem since some saturations on the input are considered.

System satisfying both Assumptions 1 and 2 is of interest since local and non-local approximations of nonlinear global dynamics may be found in the literature. For instance, in [1], local and non-local homogeneous approximations of nonlinear systems are studied. Moreover, as shown in the example introduced in Section 5.1 below, it might be useful to split a global sector condition in two pieces (a local and a non-local one) in order to get a solution where the usual LMI-based sufficient conditions obtained from [6, 12] are too conservative. Compared to the preliminary version of this paper presented in [3], two new vectors are involved in the definition of the non local sector condition (i.e. R_∞ and Q_∞). This allows to extend the class of considered systems.

In this paper, the following problem is addressed:

Problem: *Under Assumptions 1 and 2, is it possible to design a nonlinear control law $u = \alpha(x)$ where $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function ensuring global asymptotic stabilization of the origin for the system (1)?*

Before considering this global stabilization problem, each sector are considered separately in Section 2. Indeed, a local (resp. a non-local) controller $u = \alpha_0(x)$ (resp. $u = \alpha_\infty(x)$) is synthesized employing the local (resp. non-local) sector condition of Assumption 1 (resp. of Assumption 2). After this preliminary step, a new controller, which is equal to the local controller $u = \alpha_0(x)$ on a neighborhood of the origin and equal to the non-local controller $u = \alpha_\infty(x)$ outside a compact set, is designed. This construction is based on [2] and is considered in Section 3. We then formalize a sufficient condition, expressed in terms of the existence of solutions to LMIs constraints, allowing us to address the global stabilization problem in one step in Section 4. Three numerical examples illustrate the previous results in Section 5. Section 6 contains some concluding remarks.

Notation. The Euclidian norm is denoted by $|\cdot|$. For a positive real number n, I_n (resp. $0_{n,m}$) denotes the identity matrix (resp. the null matrix) in $\mathbb{R}^{n\times n}$ (resp. in $\mathbb{R}^{n\times m}$). The subscripts may be omitted when there is no ambiguity. Moreover, for a vector x the diagonal matrix defined by the entries of x is denoted $\mathtt{Diag}(x)$ while for two (or more) matrices A, B, $\mathtt{Diag}[A, B]$ is the block diagonal matrix formed by A and B. For a matrix M, $\mathtt{He}(M) = M+M'$. Finally, for each integer q, $\mathbb{1}_q$ denotes the vector in $\mathbb{R}^q$ defined by $\mathbb{1}_q = (1, \ldots, 1)'$.

2. Design of local and of non-local controllers

2.1. Local case

In this section, we consider Assumption 1 and we design a state feedback ensuring local asymptotic stabilization of the origin for the system (1). Note that if we introduce:

$$A_0 = A + GM_0 \ , \phi_0(x) = \phi(x) - M_0 x \ , S_0 = N_0 - M_0 \ ,$$

system (1) can be rewritten as:

$$\dot{x} \ = \ A_0\, x \ + \ B\, u \ + \ G\, \phi_0(x) \ , \tag{5}$$

and the local sector condition becomes

$$\phi_0(x)'\,(\phi_0(x) - S_0 x) \leq 0 \ , \ \forall \ |x| \leq v_0 \ . \tag{6}$$

Hence, inspired by [6, 12], we can state a sufficient condition to get local asymptotic stabilization of the origin:

Proposition 2.1. *Suppose Assumption 1 is satisfied (hence (6) holds). If there exist a symmetric positive definite matrix W_0 in $\mathbb{R}^{n\times n}$, two matrices H_0 in $\mathbb{R}^{m\times n}$, and J_0 in $\mathbb{R}^{m\times p}$ and a positive real number η_0 satisfying the following inequality:*

$$\begin{bmatrix} \mathtt{He}(A_0W_0 + BH_0) & \star \\ J_0'\,B' + \eta_0 G' + S_0W_0 & -2\eta_0 I_p \end{bmatrix} < 0 \ , \tag{7}$$

then the control law $u = \alpha_0(x)$ where

$$\alpha_0(x) \ = \ K_0\,x \ + \ \eta_0^{-1} J_0\,\phi_0(x) \ , \tag{8}$$

with $K_0 \ = \ H_0\,W_0^{-1}$ makes the origin of the system a locally asymptotically stable equilibrium, with basin of attraction containing the set

$$\mathcal{E}(W_0^{-1}, R_0) = \{x \in \mathbb{R}^n \ , \ x'\,W_0^{-1}\,x \leq R_0\}$$

where R_0 is any positive real number satisfying

$$R_0\,W_0 - v_0^2 I_n \leq 0 \ . \tag{9}$$

Proof: Consider the Lyapunov function candidate $V_0(x) \ = \ x'P_0x$ where $P_0 = W_0^{-1}$. The satisfaction of relation (9) means that the ellipsoid $\mathcal{E}(P_0, R_0) = \{x \in \mathbb{R}^n, x'\,P_0\,x \leq R_0\}$ is included in the ellipsoidal set $\{x \in \mathbb{R}^n, |x| \leq v_0\}$. Thus, the sector condition (3) is satisfied for any $x \in \mathcal{E}(P_0, R_0)$.

The time-derivative of V_0 along the trajectories of the system (5) with the control law (8) reads:

$$\dot{V}_0(x) = x'\,[(A_0 + BK_0)'P_0 + P_0(A_0 + BK_0)]\,x + 2x'P_0(B\,\eta_0^{-1}J_0 + G)\,\phi_0(x) \ .$$

Thus, by using the sector condition (3) and with $\eta_0 > 0$, it yields for all x in $x \in \mathcal{E}(P_0, R_0)$:

$$\dot{V}_0(x) \ \leq \ \dot{V}_0(x) \ - \ 2\eta_0^{-1}\,\phi_0(x)'\ (\phi_0(x) - S_0\,x) \ .$$

Hence, this implies, for all $x \in \mathcal{E}(P_0, R_0)$ that:

$$\dot{V}_0(x) \ \leq \ \begin{bmatrix} x' & \phi_0(x)' \end{bmatrix} \mathcal{M}_0 \begin{bmatrix} x \\ \phi_0(x) \end{bmatrix} \ ,$$

where $\mathcal{M}_0 \in \mathbb{R}^{(n+p)\times(n+p)}$ is defined by:

$$\mathcal{M}_0 = \begin{bmatrix} \mathtt{He}(P_0(A_0 + BK_0)) & \star \\ (\eta_0^{-1}J_0'B + G')P_0 + \eta_0^{-1}S_0 & -2\eta_0^{-1}I_p \end{bmatrix} \ .$$

By pre- and post-multiplying relation (7) by $\mathtt{Diag}[W_0^{-1}, \eta_0 I_p] = \mathtt{Diag}[P_0, \eta_0 I_p]$, it follows that $\mathcal{M}_0 < 0$. Hence, if relation (7) is satisfied one can conclude that $\dot{V}_0(x) < 0$, for any $x \in \mathcal{E}(P_0, R_0)$, $x \neq 0$. It follows that the origin of the system (5) closed by the control law (8) is locally asymptotically stable, and the ellipsoid $\mathcal{E}(P_0, R_0)$ is included in the basin of attraction of the origin. □

Note that the sufficient condition given by Proposition 2.1 is given in terms of solutions to linear matrix inequalities for which some powerful LMI solvers (see [17] for instance) may be used as illustrated by the numerical examples given in Section 5.

2.2. Non-local case

A result similar to Proposition 2.1 can be obtained when considering Assumption 2. Indeed, with:

$$A_\infty = A + GM_\infty \ , \ \phi_\infty(z) = \phi(x) - M_\infty x - R_\infty \ , \ S_\infty = N_\infty - M_\infty \ , \ T_\infty = R_\infty - Q_\infty \ ,$$

system (1) becomes:

$$\dot{x} \ = \ A_\infty \, x \ + \ B \, u \ + \ G \, \phi_\infty(x) \ + \ GR_\infty \ , \tag{10}$$

and the non-local sector condition (i.e. inequality (4)) yields:

$$\phi_\infty(x)' \, (\phi_\infty(x) - S_\infty x - T_\infty) \leq 0 \ , \forall \ |x| \geq v_\infty \ . \tag{11}$$

With these data, a sufficient condition to get global asymptotic stabilization of a set containing the origin can be stated:

Proposition 2.2. *Suppose Assumption 2 is satisfied (hence (11) holds). If there exists a symmetric positive definite matrix W_∞ in $\mathbb{R}^{n\times n}$, two matrices H_∞ in $\mathbb{R}^{m\times n}$, and J_∞ in $\mathbb{R}^{m\times p}$, two vectors L_∞ and $\mathbb{Z}_m$ in $\mathbb{R}^m$ and two positive real numbers η_∞ and τ_∞ satisfying the matrix inequality:*

$$\begin{bmatrix} \mathtt{He}(A_\infty W_\infty + BH_\infty) & \star & \star & \star \\ (BJ_\infty + \eta_\infty G)' + S_\infty W_\infty & -2\eta_\infty I_p & \star & \star \\ \mathtt{Diag}(L_\infty)B' + \mathbb{Z}_m R'_\infty G' & \mathbb{Z}_m T'_\infty & -\tau_\infty \frac{v_\infty^2}{m} I_m & \star \\ W_\infty & 0_{n,p} & 0_{n,m} & -\tau_\infty I_n \end{bmatrix} \ < \ 0 \ , \tag{12}$$

and the equality

$$\mathbb{Z}'_m \mathbb{1}_m = \tau_\infty \tag{13}$$

then the control law $u = \alpha_\infty(x)$ where

$$\alpha_\infty(x) \ = \ K_\infty \, x \ + \ \eta_\infty^{-1} J_\infty \, \phi_\infty(x) + \tau_\infty^{-1} L_\infty \ , \tag{14}$$

with $K_\infty = H_\infty W_\infty^{-1}$ makes the solutions of the closed-loop system complete and the set

$$\mathcal{E}(W_\infty^{-1}, r_\infty) = \{x \ \in \ \mathbb{R}^n \ , x' \, W_\infty^{-1} \, x \leq r_\infty\} \ ,$$

globally and asymptotically stable where r_∞ is any positive real number such that

$$v_\infty^2 I_n - W_\infty r_\infty \ \leq \ 0 \ . \tag{15}$$

Proof: The Lyapunov function candidate is defined as $V_\infty(x) = x' P_\infty x$, where $P_\infty = W_\infty^{-1}$ which is symmetric positive definite. The time derivative V_∞ along the trajectories of the system (5) with the control law (14) reads:

$$\dot{V}_\infty = x'[\mathtt{He}(P_\infty A_\infty + P_\infty BK_\infty)]x + 2\phi_\infty(x)'(B\eta_\infty^{-1} J_\infty + G)' P_\infty x + 2(\tau_\infty^{-1} BL_\infty + GR_\infty)' P_\infty x \ .$$

Thus, by using the non-local sector condition (3), it follows, for all x such that $|x| \geq v_\infty$,

$$\dot{V}_\infty \leq x'[\mathtt{He}(P_\infty A_\infty + P_\infty BK_\infty)]x + 2\phi_\infty(x)'(B\eta_\infty^{-1} J_\infty + G)' P_\infty x + 2\tau_\infty^{-1}(BL_\infty + \tau_\infty GR_\infty)' P_\infty x$$
$$-2 \, \eta_\infty^{-1} \phi_\infty(x)' \, (\phi_\infty(x) - S_\infty x - T_\infty) - \tau_\infty^{-1}(v_\infty^2 - x'x) \ .$$

With the equality constraint (13) it yields,

$$\dot{V}_\infty \leq x'[\mathtt{He}(P_\infty A_\infty + P_\infty B K_\infty)]x + 2\phi_\infty(x)'(\eta_\infty^{-1} B J_\infty + G)' P_\infty x$$
$$+2\tau_\infty^{-1}\left(B\mathtt{Diag}(L_\infty)\mathbb{1}_m + G R_\infty \mathbb{Z}_m' \mathbb{1}_m\right)' P_\infty x$$
$$-2\,\eta_\infty^{-1}\phi_\infty(x)'\left(\phi_\infty(x) - S_\infty x - \tau_\infty^{-1} T_\infty \mathbb{Z}_m' \mathbb{1}_m\right) - \tau_\infty^{-1}(\mathbb{1}_m' \mathbb{1}_m \tfrac{v_\infty^2}{m} - x'x)\ .$$

This can be rewritten in matrix form as,

$$\dot{V}_\infty \ \leq \ \begin{bmatrix} x \\ \phi_\infty \\ \mathbb{1}_m \end{bmatrix}' \mathcal{M}_\infty \begin{bmatrix} x \\ \phi_\infty \\ \mathbb{1}_m \end{bmatrix} \tag{16}$$

where,

$$\mathcal{M}_\infty = \begin{bmatrix} \mathtt{He}(P_\infty A_\infty + P_\infty B K_\infty) + \tau_\infty^{-1} I_n & \star & \star \\ (\eta_\infty^{-1} B J_\infty + G)' P_\infty + \eta_\infty^{-1} S_\infty & -2\eta_\infty^{-1} I_p & \star \\ \tau_\infty^{-1}(\mathtt{Diag}(L_\infty)B' + \mathbb{Z}_m R_\infty' G')P_\infty & \eta_\infty^{-1}\tau_\infty^{-1}\mathbb{Z}_m T_\infty' & -\tau_\infty^{-1}\frac{v_\infty^2}{m} I_m \end{bmatrix}.$$

The matrix $\widetilde{\mathcal{M}}_\infty$ obtained by Pre- and Post multiplying $\mathcal{M}_\infty$ by $\mathtt{Diag}[W_\infty, \eta_\infty I_p, \tau_\infty I_m]$ where $W_\infty = P_\infty^{-1}$, is defined as

$$\widetilde{\mathcal{M}}_\infty = \begin{bmatrix} \mathtt{He}(A_\infty W_\infty + B K_\infty) + \tau_\infty^{-1} W_\infty^2 & \star & \star \\ (B J_\infty + \eta_\infty G)' + S_\infty W_\infty & -2\eta_\infty I_p & \star \\ \mathtt{Diag}(L_\infty)B' + \mathbb{Z}_m R_\infty' G' & \mathbb{Z}_m T_\infty' & -\tau_\infty \frac{v_\infty^2}{m} I_m \end{bmatrix}$$

Inequality (12) with the Schur complement yields that $\widetilde{\mathcal{M}}_\infty < 0$. Hence, $\mathcal{M}_\infty < 0$ and with (16), this implies that $\dot{V}_\infty < 0$ along the trajectories of (10) as long as the trajectories remain in the set $\{x,\ |x| \geq v_\infty\}$. This implies completeness of the trajectories of system (1) closed by the control law (14). Moreover, inequality (15) yields:

$$r_\infty |x|^2 \ \geq \ v_\infty^2\, x' P_\infty x \quad , \qquad \forall\, x \ \in \ \mathbb{R}^n\ .$$

Consequently, the set $\{x,\ |x| \geq v_\infty\}$ contains the set $\mathcal{E}(P_\infty, r_\infty)$. Therefore:

$$\dot{V}_\infty(x) \ < \ 0 \quad , \quad \forall\, x \text{ such that } V_\infty(x) \geq r_\infty\ ,$$

and the set $\mathcal{E}(P_\infty, r_\infty)$ is globally asymptotically stable. □

3. Design of a globally and asymptotically stabilizing controller

In this section, it is assumed that the local stabilization problem and the non-local one have been solved following Propositions 2.1 and 2.2. Hence, the controllers α_0 and α_∞ defined by (8) and (14) respectively and the Control Lyapunov functions (CLF) $x \mapsto x'P_0x$ and $x \mapsto x'P_\infty x$ are available. The problem is to unite these two controllers to get a controller making the origin a global and asymptotic stable equilibrium.

To solve this problem, the uniting strategy introduced in [2] is employed. Following this procedure, the first step is to unite the local CLF $x \mapsto x'P_0x$ and the non-local one $x \mapsto x'P_\infty x$. The first requirement is that the two sets in which we have a stability property overlap.

Assumption 3. Covering Assumption. *There exist two positive real numbers R_0 and r_∞ such that (9) and (15) are satisfied and such that*

$$r_\infty P_0 - R_0 P_\infty \; < \; 0 \; . \tag{17}$$

In Figure 1, an illustration of the covering Assumption is presented (using the numerical values of Section 5.1). This assumption implies that we have the following inclusions:

$$\{x, |x| \leq v_\infty\} \subseteq \mathcal{E}(P_\infty, r_\infty) \subset \mathcal{E}(P_0, R_0) \subseteq \{x, |x| \leq v_0\} \; .$$

To get a global stabilizing control law, we have the following result:

Theorem 3.1. *Assume that Assumptions 1, 2 and 3 hold. If there exist two matrices K_m in $\mathbb{R}^{m\times n}$ and J_m in $\mathbb{R}^{m\times p}$, five vectors L_m, $\mathbb{Z}_{1,1}$, $\mathbb{Z}_{1,2}$, $\mathbb{Z}_{2,1}$, and $\mathbb{Z}_{2,2}$ in $\mathbb{R}^m$ and eight positive real numbers $(\mu_1, \mu_2, \theta_1, \theta_2, \nu_1, \nu_2, \nu_3, \nu_4)$ such that the following LMIs are satisfied:*

$$\begin{bmatrix} \mathtt{He}(P_0\,[A \,+\, BK_m]) - \nu_1 P_0 + \nu_2 P_\infty & \star & \star \\ (J_m' B' + G') P_0 & 0_{p\times p} & \star \\ \mathtt{Diag}(L_m)' B' P_0 & 0_{m,p} & \frac{\nu_1 R_0 - \nu_2 r_\infty}{m} I_m \end{bmatrix} \tag{18}$$

$$- \mu_1 Q_0 - Q_{\infty,1} < 0 \; ,$$

$$\begin{bmatrix} \mathtt{He}(P_\infty\,[A \,+\, BK_m]) + \nu_4 P_\infty - \nu_3 P_0 & \star & \star \\ (J_m' B' + G') P_\infty & 0_{p,p} & \star \\ \mathtt{Diag}(L_m)' B' P_\infty & 0_{m,p} & \frac{\nu_3 R_0 - \nu_4 r_\infty}{m} I_m \end{bmatrix} \tag{19}$$

$$- \mu_2 Q_0 - Q_{\infty,2} < 0 \; ,$$

where

$$\begin{aligned} Q_0 &= \begin{bmatrix} \mathtt{He}(M_0' N_0) & \star & \star \\ -(M_0 + N_0) & 2I_p & \star \\ 0_{m,n} & 0_{m,p} & 0_{m,m} \end{bmatrix} , \\ Q_{\infty,1} &= \begin{bmatrix} \theta_1 \mathtt{He}(M_\infty' N_\infty) & \star & \star \\ -\theta_1\,(M_\infty + N_\infty) & 2\theta_1 I_p & \star \\ \mathbb{Z}_{1,1}(R_\infty' N_\infty + Q_\infty' M_\infty) & -\mathbb{Z}_{1,2}(Q_\infty' + R_\infty') & \frac{2\theta_1}{m} R_\infty' Q_\infty I_m \end{bmatrix} , \\ Q_{\infty,2} &= \begin{bmatrix} \theta_2 \mathtt{He}(M_\infty' N_\infty) & \star & \star \\ -\theta_2\,(M_\infty + N_\infty) & 2\theta_2 I_p & \star \\ \mathbb{Z}_{2,1}(R_\infty' N_\infty + Q_\infty' M_\infty) & -\mathbb{Z}_{2,2}(Q_\infty' + R_\infty') & \frac{2\theta_2}{m} R_\infty' Q_\infty I_m \end{bmatrix} , \end{aligned} \tag{20}$$

with the equality constraint

$$\mathbb{Z}_{1,1}' \mathbb{1}_m = \mathbb{Z}_{1,2}' \mathbb{1}_m = \theta_1 \; , \; \mathbb{Z}_{2,1}' \mathbb{1}_m = \mathbb{Z}_{2,2}' \mathbb{1}_m = \theta_2 \; . \tag{21}$$

then there exists a continuous function $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ such that the control law $u = \alpha(x)$ makes the origin a globally asymptotically stable equilibrium for system (1).

Before proving this result, it has to be noticed that this result is not only an existence result but also a design procedure of a stabilizing controller. Indeed, a possible control law ensuring a global asymptotic stabilization of the origin of system (1) is given in [2, Theorem 3.1] and is expressed as:

$$\alpha(x) \; = \; \mathcal{H}(x) \; - \; k\, c(x) \left(\frac{\partial V}{\partial x}(x) B \right)' \tag{22}$$

with $\mathcal{H}$ a continuous function such that

$$\mathcal{H}(x) \;=\; \begin{cases} \alpha_0(x) & \text{if } V_\infty(x) \;\leq\; r_\infty \;, \\ \alpha_\infty(x) & \text{if } V_0(x) \;\geq\; R_0 \;. \end{cases}$$

The function c is any continuous function such that

$$c(x) \begin{cases} \;=\; 0 & \text{if } V_0(x) \;\geq\; R_0 \;\text{ or } V_\infty(x) \;\leq\; r_\infty \;, \\ \;>\; 0 & \text{if } V_0(x) \;<\; R_0 \;\text{ and } V_\infty(x) \;>\; r_\infty \;. \end{cases}$$

k is a positive real number sufficiently large and finally V is a global CLF for system (1) obtained following the procedure introduced in [2, Theorem 2.1] which enables to unite both CLFs V_0 and V_∞.

Proof: To prove Theorem 3.1, we apply the procedure introduced in [2, Theorem 2.1] which is based on the uniting of a local and a non-local control Lyapunov function.

First, we introduce both functions $V_0(x) = x'P_0x$ and $V_\infty(x) \;=\; x'P_\infty x$. Note that with Assumption 3, we get that both functions V_0 and V_∞ and both positive real numbers R_0 and r_∞ satisfy [2, Assumption 1]. More precisely, P_0 and P_∞ being solutions of Propositions 2.1 and 2.2, we get that the function V_0 satisfies, for all x in $\{x, V_0(x) \leq R_0\}$,

$$\frac{\partial V_0}{\partial x}(x)[Ax + B\alpha_0(x) + G\phi(Lx)] < 0 \;,$$

and V_∞ satisfies, for all x in $\{x, V_\infty(x) \geq r_\infty\}$,

$$\frac{\partial V_\infty}{\partial x}(x)[Ax + B\alpha_\infty(x) + G\phi(Lx)] < 0 \;.$$

Finally, with the covering assumption, the functions V_0 and V_∞ satisfy

$$\{x, V_\infty(x) > r_\infty\} \cup \{x, V_0(x) < R_0\} = \mathbb{R}^n \;.$$

and the set $\{x, V_0(x) \leq R_0\} \cup \{x, V_\infty(x) \geq r_\infty\}$ is a non empty compact subset of $\mathbb{R}^n$.

Moreover, setting $u = K_m\, x \;+\; J_m\phi(x) \;+\; L_m$ yields along the trajectories of the system (1),

$$\dot{V}_0(x) = x'\left[P_0\left(A \;+\; BK_m\right) \;+\; (A \;+\; BK_m)'P_0\right] x \\ +2\, x'P_0(G + BJ_m)\,\phi(x) + 2\, x'P_0B\,\mathtt{Diag}(L_m)\mathbb{1}_m \;.$$

This inequality can be rewritten in matrix form as:

$$\dot{V}_0(x) = \left[\begin{array}{ccc} x' & \phi(x)' & \mathbb{1}_m' \end{array}\right] \left[\begin{array}{ccc} \mathtt{He}(P_0\left[A \;+\; BK_m\right]) & \star & \star \\ (J_m'B' + G')P_0 & 0 & \star \\ \mathtt{Diag}(L_m)'B'P_0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x \\ \phi(x) \\ \mathbb{1}_m \end{array}\right] . \tag{23}$$

Note that, for all x such that $\{x, V_0(x) \leq R_0\} \cup \{x, V_\infty(x) \geq r_\infty\}$:

$$\left[\begin{array}{ccc} x' & \phi(x)' & \mathbb{1}_m' \end{array}\right] \left[\begin{array}{ccc} \nu_1P_0 - \nu_2P_\infty & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & \frac{\nu_2 r_\infty - \nu_1 R_0}{m} I_m \end{array}\right] \left[\begin{array}{c} x \\ \phi(x) \\ \mathbb{1}_m \end{array}\right] < 0 \;, \tag{24}$$

where (ν_1, ν_2) are the positive real numbers given in Theorem 3.1. Moreover, with (9) and (15) we get that all x in the set $\{x, V_0(x) \leq R_0\} \cup \{x, V_\infty(x) \geq r_\infty\}$ are in the set $\{x,\; v_\infty \leq |x| \leq v_0\}$.

Hence, using both inequalities (3) and (4), we get, for all x in the set $\{x, V_0(x) \leq R_0\} \cup \{x, V_\infty(x) \geq r_\infty\}$,

$$\left[\begin{array}{ccc} x' & \phi(x)' & \mathbb{1}_m' \end{array}\right] [\mu_1 Q_0 + Q_{\infty,1}] \left[\begin{array}{c} x \\ \phi(x) \\ \mathbb{1}_m \end{array}\right] < 0 \tag{25}$$

where Q_0 and $Q_{\infty,1}$ are given in (20). Consequently, with inequalities (23), (24) and (25) and the property (18), it yields that

$$\dot{V}_0(x) < 0 \ , \ \forall x \in \{x, V_0(x) \leq R_0, V_\infty(x) \geq r_\infty\}$$

With (19) the same conclusion holds for V_∞, i.e.:

$$\dot{V}_\infty(x) < 0 \ , \ \forall x \in \{x, V_0(x) \leq R_0, V_\infty(x) \geq r_\infty\}$$

Consequently, a same control can be designed for both functions V_0 and V_∞ for each x in $\{x, V_0(x) \leq R_0, V_\infty(x) \geq r_\infty\}$. With [2, Proposition 2.2], we get that [2, Assumption 2] is also satisfied. Consequently all Assumptions of [2, Theorem 3.1] are satisfied and it yields that there exists a control law ensuring global asymptotic stabilization of the origin of system (1). □

From the previous results, a design strategy to get a stabilizing control law for system (1) may be described by the following algorithm:

Design separately a local and a non-local CLF (i.e. $P_0 = W_0^{-1}$ and $P_\infty = W_\infty^{-1}$) via the LMIs (7) and (12), and check if

1. *they satisfy the covering Assumption (17) with R_0 and r_∞ satisfying (9) and (15);*
2. *they satisfy the LMIs feasibility conditions (18) and (19) to be united.*

If these two tests are positive then build stabilizing control law given by (22).

4. Design in one step

In this section, we investigate the possibility of solving the design problem in one shot. In other words, we wish to find an LMI formulation to prove the existence of matrices P_0 and P_∞ satisfying the conditions in items 1) and 2) of the previous algorithm.

4.1. About the covering assumption

Assumption 3 may fail when considering an arbitrary pair of matrices P_0 and P_∞ computed using Propositions 2.1 and 2.2.

Moreover, note that inequality (17), combined with Propositions 2.1 and 2.2, is not linear in P_0 or P_∞ since R_0 and r_∞ depend on P_0 and P_∞ through the constraints (9) and (15) in which $W_0 = P_0^{-1}$ and $W_\infty = P_\infty^{-1}$.

Nevertheless, note that, when $R_0 = r_\infty = \frac{1}{\rho}$, the covering Assumption can be easily defined as the following LMI:

$$W_0 - W_\infty > \ 0 \ . \tag{26}$$

Moreover, note that the two matrix inequality constraints (9) and (15) can be recast as the following LMI constraints:

$$\rho v_\infty^2 I_n - W_\infty \leq 0 \ , \ W_0 - \rho v_0^2 I_n \leq 0 \tag{27}$$

Consequently, this feasibility constraint can be added easily in the design of W_0 and W_∞ (i.e. of P_0 and P_∞).

To summarize, we have:

Proposition 4.1. *Suppose there exist two positive definite matrices W_0, W_∞ in $\mathbb{R}^{n\times n}$ and a real number $\rho > 0$ such that inequalities (26) and (27) are satisfied, then the covering assumption (i.e. inequality (17)) is also satisfied with $R_0 = r_\infty = \frac{1}{\rho}$.*

4.2. About the second feasibility constraint

In this section, in order to ease the presentation, it is assumed that the two matrices Q_∞ and R_∞ which appear in the definition of the non-local sector condition are equal to zero. However, all the following results could be extended to the more general case.

To include the feasibility constraints (18) and (19) into the design of a global asymptotic stabilizer, we need to restrict ourselves to a specific class of matrices W_0, H_0, J_0, W_∞, H_∞, and J_∞ solutions of (7), (12), (26), (27).

To be more precise we consider the subclass of solutions such that the conditions (18) and (19) are satisfied, by particularizing these conditions as LMI conditions. To do that we use elimination lemma [16], and we get the following result.

Proposition 4.2. *If the local and the non-local conditions (3) and (4) are such that $N_0 = N_\infty = 0$ and $Q_\infty = R_\infty = 0$ and if there exist two symmetric positive definite matrices W_0 and W_∞ in $\mathbb{R}^{n\times n}$, two matrices J_m in $\mathbb{R}^{m\times p}$ and H_m in $\mathbb{R}^{m\times n}$, an invertible matrix F in $\mathbb{R}^{n\times n}$, satisfying the LMI constraints*

$$\begin{bmatrix} \mathtt{He}(AF+BH_m) & \star & \star \\ -W_0+F+AF+BH_m & -2W_0 & BJ_m+G-W_0G \\ (M_0+M_\infty)F+(BJ_m+G)'+AF+BH_m & \star & -4I_p-\mathtt{He}(G'(BJ_m+G)) \end{bmatrix} < 0 \tag{28}$$

$$\begin{bmatrix} \mathtt{He}(AF+BH_m) & \star & \star \\ -W_\infty+F+AF+BH_m & -2W_\infty & BJ_m+G-W_\infty G \\ (M_0+M_\infty)F+(BJ_m+G)'+G'(AF+BH_m) & \star & -4I_p-\mathtt{He}(G'(BJ_m+G)) \end{bmatrix} < 0 \tag{29}$$

then there exist positive real numbers ν_1, ν_2, ν_3 ν_4, and vectors $\mathbb{Z}_{1,1}$, $\mathbb{Z}_{1,2}$, $\mathbb{Z}_{2,1}$ and $\mathbb{Z}_{2,2}$ in $\mathbb{R}^m$ such that inequalities (18) and (19) and the equality (21) hold with $P_0 = W_0^{-1}$, $P_\infty = W_\infty^{-1}$, $K_m = H_m F^{-1}$, $\mu_1 = \mu_2 = \theta_1 = \theta_2 = 1$, and $L_m = 0_{m,1}$.

Proof: First note that Pre- and Post- multiplying inequality (28) respectively by $\mathtt{Diag}\,[F_1, P_0, I_p]$ and $\mathtt{Diag}\,[F_1', P_0, I_p]$ with $F_1' = F^{-1}$ gives the following inequality.

$$\begin{bmatrix} \mathtt{He}(F_1(A+BK_m)) & \star & \star \\ P_0-F_1'+P_0(A+BK_m) & -2P_0 & P_0(BJ_m+G)-G \\ M_0+M_\infty+(J_m'B'+G')F_1'+G'(A+BK_m) & \star & -4I_p-\mathtt{He}(G'(BJ_m+G)) \end{bmatrix} < 0$$

This inequality, can be rewritten as

$$\mathfrak{N}_0 + \begin{bmatrix} F_1 \\ P_0 \\ G' \end{bmatrix} \begin{bmatrix} A+BK_m & -I_n & BJ_m+G \end{bmatrix} + \begin{bmatrix} A'+K_m'B' \\ -I_n \\ J_m'B'+G' \end{bmatrix} \begin{bmatrix} F_1' & P_0 & G \end{bmatrix} < 0\ ,$$

with,

$$\mathfrak{N}_0 = \begin{bmatrix} 0 & \star & \star \\ P_0 & 0 & \star \\ M_0 + M_\infty & 0 & -4I_p \end{bmatrix} .$$

This implies that

$$\begin{bmatrix} a' & b' & c' \end{bmatrix} \mathfrak{N}_0 \begin{bmatrix} a \\ b \\ c \end{bmatrix} < 0$$

for all (a, b, c) in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$ such that

$$\begin{bmatrix} A + BK_m & -I_n & BJ_m + G \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 . \tag{30}$$

Note that setting the matrix in $\mathbb{R}^{(n+p)\times(2n+p)}$:

$$\mathfrak{K}_0 = \begin{bmatrix} I_n & 0 \\ A + BK_m & BJ_m + G \\ 0 & I_n \end{bmatrix} ,$$

the set of points (a, b, c) in $\mathbb{R}^{2n+p}$ satisfying the constraint (30) is equal to the image of the matrix $\mathfrak{K}_0$. Consequently, we get

$$\mathfrak{K}_0' \mathfrak{N}_0 \mathfrak{K}_0 < 0 .$$

Let ν_1 and ν_2 be such that

$$\mathfrak{K}_0' \mathfrak{N}_0 \mathfrak{K}_0 + \begin{bmatrix} -\nu_1 P_0 + \nu_2 P_\infty & \star \\ 0_{n\times p} & 0_{p\times p} \end{bmatrix} < 0 , \quad \frac{\nu_1 R_0 - \nu_2 r_\infty}{m} < 0 .$$

Note that this choice is always possible taking for instance $\nu_1 = 0$ and ν_2 small. This implies that we get

$$\begin{bmatrix} \mathtt{He}(P_0 [A + BK_m]) - \nu_1 P_0 + \nu_2 P_\infty & \star & \star \\ (J_m' B' + G') P_0 + M_0 + M_\infty & -4I_p & \star \\ 0_{n,m} & 0_{p,m} & \frac{\nu_1 R_0 - \nu_2 r_\infty}{m} I_m \end{bmatrix} < 0 .$$

This matrix inequality is exactly (18) in the particular case where $\mu_1 = \theta_1 = 1$, $N_0 = N_\infty = Q_\infty = R_\infty = 0$ and with $L_m = 0_{m,1}$. The proof that inequality (19) holds with $\mu_2 = \theta_2 = 1$ follows the same line. Note that in the particular case where $Q_\infty = R_\infty = 0$, the only constraint we have on the matrices $\mathbb{Z}_{1,1}$, $\mathbb{Z}_{1,2}$, $\mathbb{Z}_{2,1}$ and $\mathbb{Z}_{2,2}$ is the equality constraint (21) which has always a solution (for instance $\mathbb{Z}_{1,1} = \mathbb{Z}_{1,2} = \mathbb{Z}_{2,1} = \mathbb{Z}_{2,2} \frac{1}{m} \mathbb{1}_m$). □

Note that employing a change of matrix similar to the one introduced in Section 2.1 and 2.2, we can also deal with the case where $N_0 = N_\infty \neq 0$.

The key point of the previous result is that the constraints (28) and (29) obtained are linear in the unknowns W_0 and W_∞. Consequently with Propositions 4.1 we are able to give a complete LMI formulation allowing to design a state feedback control law for system (1) making the origin a globally and asymptotically stable equilibrium. This can be summarized as follows.

Theorem 4.3. *If the local and the non-local conditions (3) and (4) are such that $N_0 = N_\infty = 0$, $Q_\infty = R_\infty = 0$ and if there exist two symmetric positive definite matrices W_0 and W_∞ in $\mathbb{R}^{n\times n}$, three matrices J_0, J_∞ and J_m in $\mathbb{R}^{m\times p}$ three matrices H_0, H_∞ and H_m in $\mathbb{R}^{m\times n}$, an invertible matrix F in $\mathbb{R}^{n\times n}$, and a real number $\rho > 0$ satisfying the matrix inequality (7), (12), (26), (27), (28) and (29), then the control law $u = \alpha(x)$ with α defined in (22) makes the origin a globally asymptotically stable equilibrium for system (1).*

Proof: This proof is a direct consequence of Propositions 4.1 and 4.2 and Theorem 3.1. □

5. Numerical examples

To illustrate the proposed procedure and to motivate it, three different examples are presented. The first one is an illustration of the proposed methodology to design a global stabilizing state feedback for a system satisfying Assumptions 1 and 2. The second one exhibits the interest of splitting in two pieces a global sector condition. The third one is an illustration of Theorem 4.3.

5.1. Example 1: illustration of the methodology

In this section, we illustrate the design procedure by applying Theorem 3.1. Consider system (1) described by the following data:

$$\dot{x} \;=\; Ax \;+\; B\,u \;+\; G\phi(x) \quad , \qquad x \;\in\; \mathbb{R}^2 \;, \tag{31}$$

with $A = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \end{bmatrix}'$, $G = I_2$, ϕ is the nonlinear function defined as,

$$\phi(x) = \frac{1}{2}[\lambda(|x|)(N_\infty x + Q_\infty) + (1 - \lambda(|x|))N_0 x]$$

with

$$\lambda(s) \begin{cases} = 10^{-3}s \;,\; s \le v_0 \\ = 20s - \frac{10^{-3}v0}{20} \;,\; s \in \left[v_0, \frac{20+10^{-3}v0}{20^2}\right) \\ = 1 \;,\; s \in \left[\frac{20+10^{-3}v0}{20^2}, +\infty\right) \end{cases}$$

and T_∞, S_∞ and S_0 are the matrices defined as $Q_\infty = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$, $N_\infty = 4I_2$, $N_0 = 0.1I_2$.

It can be checked that this function satisfies Assumptions 1 and 2 with the data $v_0 = 14$ and $v_\infty = 2$, $M_0 = M_\infty = 0$ and $R_\infty = 0$. Applying Propositions 2.1 and 2.2, we may compute the controllers $\alpha_0 :\; x \mapsto K_0 x + \eta_0^{-1} J_0 \phi(x)$ and $\alpha_\infty :\; x \mapsto K_\infty x + \eta_\infty^{-1} J_\infty \phi(x) + \tau_\infty L_\infty$ with

$$\begin{array}{c} K_0 \;=\; \begin{bmatrix} -5.804 & -1.2312 \end{bmatrix} \;,\; J_0 \;=\; \begin{bmatrix} 0.01298 & -0.54668 \end{bmatrix} \;,\; \eta_0 \;=\; 0.44488 \;, \\ K_\infty \;=\; \begin{bmatrix} -76.3638 & -29.2069 \end{bmatrix} \;,\; J_\infty \;=\; \begin{bmatrix} 0.4416 & -2.3716 \end{bmatrix} \;,\; \eta_\infty \;=\; 0.93133 \;, \\ \tau_\infty \;=\; 0.08964 \;,\; L_\infty \;=\; 0.52718 \end{array}$$

such that the conclusion of Propositions 2.1 and 2.2 holds with

$$P_0 \;=\; \begin{bmatrix} 1.1262 & 0.1436 \\ 0.1436 & 1.0006 \end{bmatrix} \;,\; R_0 \;=\; 177.7115 \;,\; P_\infty \;=\; \begin{bmatrix} 46.3220 & 15.7388 \\ 15.7388 & 8.1695 \end{bmatrix} \;,\; r_\infty \;=\; 183.8474$$

Figures 2 and 3 show the behavior of the closed-loop system with the local controller and the non-local one obtained respectively from Propositions 2.1 and 2.2. Note that with the local controller the basin of attraction is a compact subset and that, with the non-local one, the trajectories do not converge toward the origin (but to another equilibrium).

The covering assumption, namely Assumption 3, is satisfied. This fact is depicted in Figure 1. Moreover Theorem 3.1 applies and the controller (22) makes the origin a global asymptotically stable equilibrium with parameter $k = 1$ and the functions c is given as:

$$c(x) = \frac{1}{1 + \left|\frac{\partial V}{\partial x}(x)B\right|} \max(0, \min((R_0 - x'P_0x)(x'P_\infty x - r_\infty), 1));$$

where V is obtained from [2, Equation (8)] with the parameters $R_0 = 190$ and $r_\infty = 150$. Figure 4 is the time-evolution of a trajectory where it can be checked that the solution converges toward the origin with the uniting controller.

5.2. Example 2: Necessity to split in two sector conditions

In this section, we check on an example that it may be necessary to split a sector condition in two sector conditions, and we illustrate the design procedure by applying Theorem 3.1. In this example, system (1) is given with the following data:

$$\dot{x} \;=\; B\,u \;+\; G\phi(x) \quad , \qquad x \;\in\; \mathbb{R}^3 \;, \tag{32}$$

with $B = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}'$, $G = I_3$. To complete the definition of system (1), it remains to introduce the function $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ representing the nonlinearity of the system. First consider both matrices M_∞ in $\mathbb{R}^{3\times 3}$ and N_0 in $\mathbb{R}^{3\times 3}$ defined as

$$M_\infty = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad , \qquad N_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \;.$$

The nonlinear function ϕ is defined as a locally Lipschitz continuous path interpolating M_∞ and N_0, i.e[5]:

$$\phi(x) = \lambda(|x|)M_\infty x + (1 - \lambda(|x|))N_0 x \;,\; x \in \mathbb{R}^3 \;, \tag{33}$$

where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing locally Lipschitz function such that[6]:

$$\lambda(0) \;=\; 0 \quad , \qquad \lim_{s\to+\infty} \lambda(s) \;=\; 1 \;.$$

We wish to find a controller guaranteeing global asymptotic stabilization of the origin.

5.2.1. Employing a unique global sector condition

A first strategy to address the stabilization problem for system (32) is to check if the solvability conditions inspired by [6, 12] are satisfied. For this purpose, let us first prove that the nonlinear function ϕ satisfies a (global) sector condition:[7]

[5]This is typically the case when a system is modeled as an interpolation of several linear systems.

[6]For instance λ can be defined by $\lambda(s) = \frac{2}{\pi}\arctan(s)$ for all $s \geq 0$.

[7]Note that the proof of Proposition 5.1 does not use the expression of the system (1) but only properties of the nonlinearity ϕ as introduced in (33).

Proposition 5.1. *The function ϕ defined in (33) satisfies the sector condition:*

$$(\phi(x) - M_\infty\, x)'(\phi(x) \;-\; N_0\, x) \;\leq\; 0 \quad , \qquad \forall\, x \;\in\; \mathbb{R}^3 \;.$$

Proof: Note that we have:

$$\begin{aligned}
&[\phi(x) - M_\infty x]'[\phi(x) - N_0 x] \\
&\quad = [\lambda(|x|)M_\infty x + (1-\lambda(|x|))N_0 x - M_\infty x]' + [\lambda(|x|)M_\infty x + (1-\lambda(|x|))N_0 x - N_0 x] \;, \\
&\quad = (1-\lambda(|x|))\lambda(|x|)[(N_0 - M_\infty)x]'[(M_\infty - N_0)x] \;, \\
&\quad = -(1-\lambda(|x|))\,\lambda(|x|)\,|(M_\infty - N_0)\,x|^2 \;, \\
&\quad \leq 0 \;.
\end{aligned}$$

This concludes the proof of Proposition 5.1. □

So, following the computations of (the proof of) Proposition 2.1, we may try to compute a global nonlinear feedback law by solving the following LMI optimization problem:

Proposition 5.2. *If there exist a symmetric positive definite matrices W in $\mathbb{R}^{3\times 3}$, and two matrices H in $\mathbb{R}^{1\times 3}$, J in $\mathbb{R}$ satisfying the LMI:*

$$\begin{bmatrix} \mathtt{He}(M_\infty W + BH) & \star \\ J'B' + I_3 + (N_0 - M_\infty)W & -2I_3 \end{bmatrix} < 0 \;,$$

then the control law:

$$u(x) \;=\; K\,x \;+\; J\,\phi(x) \;, \tag{34}$$

with $K = HW^{-1}$ makes the origin a globally and asymptotically stable equilibrium for system (32).

Using parser YALMIP [11] and LMI solver SeDuMi [17], this problem is found to be unfeasible on this particular instance. This approach is clearly too conservative and cannot be used in this specific case. This motivates to split the sector condition in two different sector conditions as done in the following section.

5.2.2. Employing the uniting controller approach

Note however that our uniting controller provides another approach to solve this stabilizing problem. First we show that the function ϕ introduced in (33) fits in the context of Assumptions 1 and 2.

Proposition 5.3. *Given two positive real numbers $0 < \lambda_\infty < \lambda_0 < 1$, Assumptions 1 and 2 are satisfied with $v_0 = \lambda^{-1}(\lambda_0)$, $M_0 = \lambda_0 M_\infty + (1-\lambda_0)N_0$, $v_\infty = \lambda^{-1}(\lambda_\infty)$, and $N_\infty = \lambda_\infty M_\infty + (1-\lambda_\infty)N_0$.*

Proof: The proof is similar to the proof of Proposition 5.2. First note that the function λ being strictly increasing, we have:

$$\{x, |x| \;\leq\; v_0\} \;=\; \{x, \lambda(|x|) \;\leq\; \lambda_0\} \;. \tag{35}$$

Moreover, we have

$$
\begin{aligned}
[\phi(x) - M_0 x]'[\phi(x) - N_0 x] &= [(\lambda(|x|) - \lambda_0) M_\infty x + (\lambda_0 - \lambda(|x|)) N_0 x)x]' \lambda(|x|) \left[(M_\infty \; - \; N_0)\, x\right] \\
&= (\lambda_0 - \lambda(|x|))\lambda(|x|)[(N_0 - M_\infty)x]'[(M_\infty - N_0)x] \; , \\
&= -(\lambda_0 - \lambda(|x|))\, \lambda(|x|)\, |(M_\infty - N_0)\, x|^2 \; ,
\end{aligned}
$$

and consequently, with (35), for all x such that $\{x, |x| \; \leq \; v_0\}$, we have

$$
[\phi(x) - M_0\, x]'[\phi(x) \; - \; N_0\, x] \leq 0 \; ,
$$

and Assumption 1 is satisfied. Similarly, we show that Assumption 2 is satisfied. □

Consequently, we are in the framework of the uniting sector condition. We follow the procedure developed in Sections 3 and 4.1, and we apply Propositions 2.1, 2.2, and 4.1. Therefore we have to check if the LMIs (7), (12), (26) and (27) have a solution in W_0, W_∞, and ρ (among others variables), and we get that Assumption 3 holds for $P_0 = W_0^{-1}$, $P_\infty = W_\infty^{-1}$, and $R_0 = r_\infty = \frac{1}{\rho}$.

Choosing $\lambda_0 = 0.6$ and $\lambda_\infty = 0.4$, and assuming that λ is a continuous function such that:

$$
v_0 = \lambda^{-1}(0.6) = 10 \; , \; v_\infty = \lambda^{-1}(0.4) = 1.5 \; ,
$$

we get the following solutions

$$
P_0 = \begin{bmatrix} 0.3635 & 0.7820 & 0.6798 \\ 0.7820 & 3.4710 & 2.7149 \\ 0.6798 & 2.7149 & 2.6948 \end{bmatrix} \; , P_\infty = \begin{bmatrix} 0.4085 & 0.8650 & 0.8252 \\ 0.8650 & 3.7691 & 3.0466 \\ 0.8252 & 3.0466 & 3.7480 \end{bmatrix} \; ,
$$

and $\rho^{-1} = R_0 = r_\infty = 16.3666$ of the LMIs (7), (12), (26) and (27). The fact that the covering Assumption is satisfied for the matrix P_0 and P_∞ with the positive real number $v_0 = 10$ and $v_\infty = 1.5$ is guaranteed by Proposition 4.1.

Hence we are in the context of Theorem 3.1 and we can check that, for the two previous matrices P_0 and P_∞, there exist two matrices J_m, K_m and four scalars μ_i satisfying the sufficient conditions (18) and (19). This is indeed the case with

$$
J_m = [-1.0000 \; -5.5737 \; -5.3139] \; , \; K_m = [1.0491 \; -0.7931 \; -0.6987].
$$

Consequently, the conclusion of Theorem 3.1 holds, we get that the control law (22) makes the origin a globally and asymptotic equilibrium for the system (32).

5.3. Example 3: Design of a stabilizing controller in one step

In this section, we apply Theorem 4.3 and we design a stabilizing controller in one step. To do that, let us consider now a second order system defined as:

$$
\left\{ \begin{array}{rcl} \dot{x}_1 & = & x_2 \; + \; x_1 \eta_1(x) \; , \\ \dot{x}_2 & = & u \; + \; x_2 \eta_2(x) \; , \end{array} \right. \tag{36}
$$

where the functions η_1 and η_2 are any continuous functions such that:

$$
\eta_1(x) \left\{ \begin{array}{l} \leq 1 \text{ if } |x| \leq v_0 \\ \leq 2 \text{ if } |x| \geq v_0 \end{array} \right. , \eta_2(x) \left\{ \begin{array}{l} \leq 1 \text{ if } |x| \leq v_\infty \\ \leq 0.5 \text{ if } |x| \geq v_\infty \end{array} \right. .
$$

Note that this system can be written in the form (1) with: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \end{bmatrix}'$, with $G = L = I_2$, and with the function $\phi(x) = \begin{bmatrix} x_1\eta_1(x) \\ x_2\eta_2(x) \end{bmatrix}$. The function $\phi(x)$ satisfies Assumptions 1 and 2 with $N_0 = N_\infty = 0$, $M_0 = I_2$, $M_\infty = \texttt{Diag}[2, 0.5]$.

Hence we are in the context of Theorem 4.3. Employing YALMIP [11] with the solver SEDUMI [17], we check the solvability of LMIs (7), (12), (26), (27), (28), and (29). These LMIs are solvable with:

$$W_0 = \begin{bmatrix} 0,6483 & -1.0998 \\ -1.0998 & 4.3131 \end{bmatrix}, W_\infty = \begin{bmatrix} 0.3610 & -0.8647 \\ -0.8647 & 3.3607 \end{bmatrix}, J_0 = \begin{bmatrix} -1.0998 & 3.3131 \end{bmatrix},$$

$$J_\infty = \begin{bmatrix} -1.7293 & 0.6804 \end{bmatrix}, J_m = \begin{bmatrix} -1.2871 & 0.0213 \end{bmatrix}, H_0 = \begin{bmatrix} -2.1135 & -6.0105 \end{bmatrix},$$

$$H_\infty = \begin{bmatrix} -1.1991 & -3.3778 \end{bmatrix}, H_m = \begin{bmatrix} 0.8534 & -3.6833 \end{bmatrix}, F = \begin{bmatrix} 0.7296 & -0.4451 \\ -0.9518 & 1.2088 \end{bmatrix},$$

and finally $\rho = 0.0681$. Consequently, the conclusion of Theorem 4.3 holds and we obtain a control law making the origin a globally and asymptotic stable equilibrium for the system (36).

6. Conclusion

In this paper we have introduced the synthesis problem of a nonlinear feedback law for a class of control systems. The control systems under consideration are those with a nonlinearity satisfying a sector condition when the state is close to the equilibrium and a (maybe) different sector condition when the state is distant from the equilibrium. We noted that encompassing both sector conditions into a unique global one may lead to a too conservative synthesis problem. This motivates us to consider both properties of the nonlinearities separately and to design successively 1) a local asymptotic stabilizing nonlinear controller whose basin of attraction contains some compact set and 2) a non-local controller which makes the previous compact set globally attractive. Then we compute a nonlinear controller which pieces together the local controller with the non-local one, and we obtain a global asymptotic stabilizing controller. We emphasize that the sufficient conditions to solve this design problem are written in terms of LMIs. Three numerical examples motivate and illustrate this approach.

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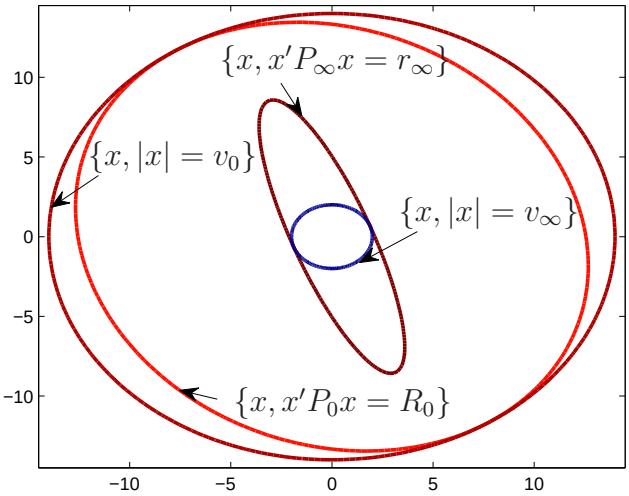

Figure 1: Illustration of the Covering Assumption for numerical example 2

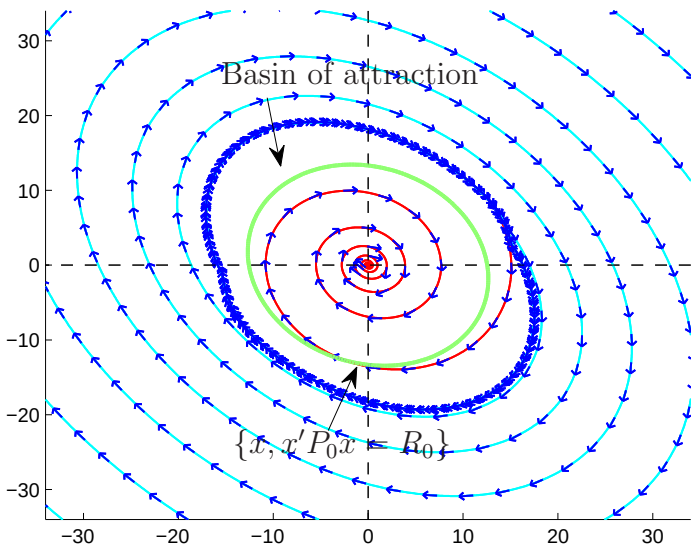

Figure 2: Illustration of the local controller

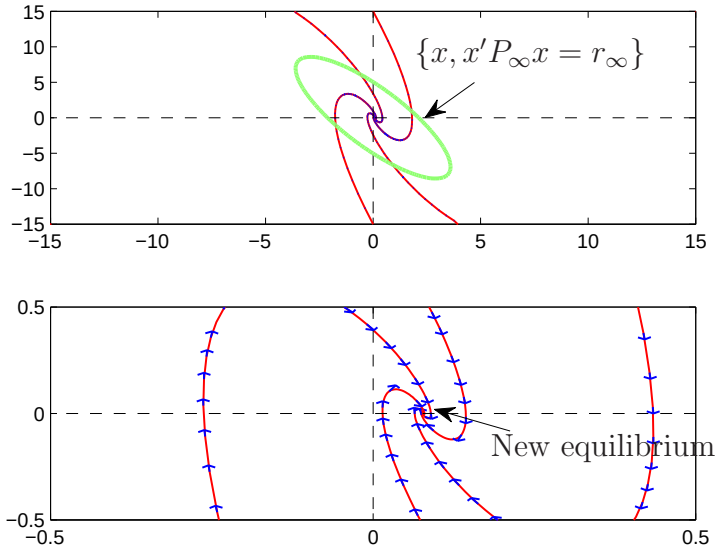

Figure 3: Illustration of the non-local controller

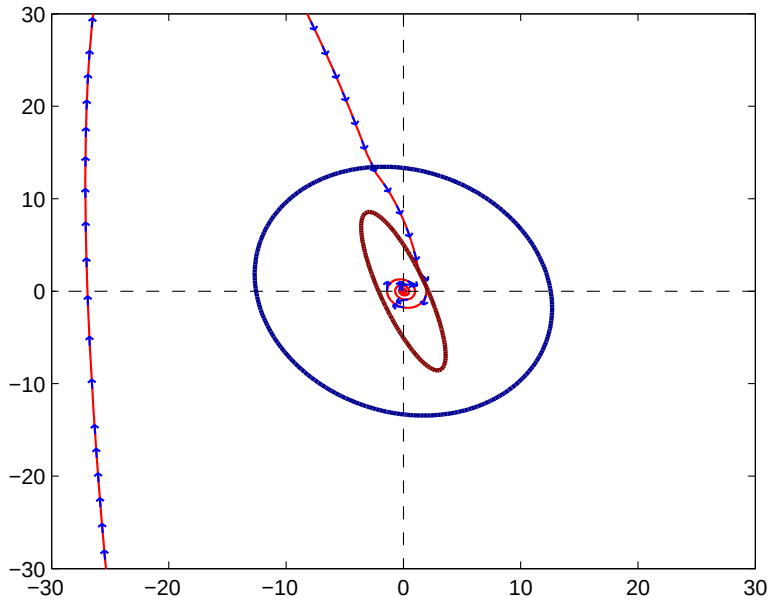

Figure 4: The uniting global controller