Further Results on Periodically Time-Varying Dynamical State-Feedback Controller Synthesis for Discrete-Time Linear Systems

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Abstract– In this paper, we enhance the quality of our preceding results on periodically time-varying dynamical state-feedback controller (PTVDSFC) synthesis for discrete-time linear periodic/time-invariant systems. We firstly revisit PTVDSFC synthesis for uncertainty-free systems and derive necessary and sufficient LMI conditions for the existence of the desired PTVDSFCs. Based on these LMIs, we next consider robust H_∞-PTVDSFC synthesis for polytopic-type uncertain systems and demonstrate that we can indeed obtain less conservative results. We finally derive a viable test to verify that the designed robust PTVDSFC is "exact" in the sense that it attains the best achievable robust H_∞ performance. This exactness verification test works fine in practice, and we show via numerical examples that exact robust control is indeed possible via PTVDSFC, even for those problems where the standard static state-feedback fails.

Key Words: Periodically time-varying dynamical controller, robust control, linear matrix inequality.

1 Introduction

During the last two decades, we have experienced drastic theoretical advances in LMI-based linear control system analysis and synthesis [7], [16], [23]. Even for those "hard" problems for which definitive analytic solutions are not yet available, we can often make successful approaches using LMIs. One of such representative topics should be the real μ analysis and synthesis, i.e., robustness analysis and synthesis of LTI systems affected by parametric uncertainties [3].

Unfortunately, however, there is a considerable gap between recent achievements on the LMI-based analysis and synthesis. Roughly speaking, robustness analysis problems are naturally formulated as robust LMI problems for which effective asymptotically exact LMI relaxation was recently established [21], [22]. However, similar robust LMI reformulations do not follow for robust control problems since they are non-convex in nature. Namely, we need another effort to get around the difficulty arising from the non-convexity.

Under this difficult situation, we made continuing efforts towards establishing a control strategy that is seemingly redundant but is indeed effective for robust control, while preserving the convexity of the corresponding synthesis problems. Partly motivated from the structural properties of existing LMIs for discrete-time system synthesis [8], [10], [1], [15], and partly motivated from the fact that periodic controllers are superior to time-invariant ones for a large class of robustness problems [18], in [11], [12], we proposed to design periodically time-varying dynamical state-feedback controllers (PTVDSFCs). In contrast to the standard static state-feedback, the PTVDSFC keeps in memory those previous states located in the same period and uses them to construct control inputs. By carefully examining this particular controller structure, we showed in [11], [12] that the PTVDSFC synthesis problem can be reduced into an LMI problem in such a sound way that it encompasses those preceding results [8], [10], [1], [15]. A notable property is that, when we deal with robust controller synthesis problems for polytopic-type uncertain LTI systems, we can gradually reduce conservatism by designing PTVDSFCs and increasing its period [11], [12].

The goal of this paper is to enhance the quality of our preceding results on the LMI-based PTVDSFC synthesis. Novel contributions that distinguish the present paper from [11], [12] can be summarized as follows: (a) We derive necessary and sufficient LMI conditions for the existence of the desired PTVDSFCs for uncertainty-free periodic/time-invariant systems. The key observations is that the *block-diagonal* structure imposed on the additional variables in the LMIs in [11], [12] can be relaxed to *block-upper-triangular*, while preserving the applicability of the linearizing change of variables for deriving LMIs . (b) Based on these sound LMIs, we next consider robust H_∞-PTVDSFC synthesis for uncertain periodic/time-invariant systems and demonstrate that we can indeed obtain less conservative results than [11], [12]. (c) To establish a more rigorous robust PTVDSFC synthesis strategy, we derive a viable test to verify that the designed PTVDSFC is "exact" in the sense that it attains the best achievable robust H_∞ performance. This exactness verification test relies on the duality theory [2], and can be regarded as the extension of those recently developed in the robustness analysis of uncertain linear systems [21], [22], [13].

We believe that the exactness verification of the designed robust controller is a striking contribution of the present paper. Interestingly enough, we illustrate via numerical examples that the exact robust control is indeed possible via PTVDSFCs, even for those problems where the standard static state-feedback fails.

We use the following notations in this paper. For given two integers k and N, we denote by $\lceil k \rceil_N$ the remainder of k divided by N. The symbols $\mathbf{1}_n$ and $\mathbf{0}_{n,m}$ stand for the identity and zero matrices of the size n and $n \times m$, respectively. We omit the size if it

is clear from context. For a real square matrix A, we define $\mathrm{He}\{A\} := A + A^T$. Finally, we denote by $\mathbf{S}_n$ and $\mathbf{P}_n$ the set of symmetric matrices and positive-definite symmetric matrices of the size n, respectively.

2 Particularly Structured Periodic System and Novel LMI Conditions

2.1 Review of Preceding Results in [11], [12]

In this subsection, we quickly review our preceding results on the analysis of particularly structured periodic systems [11], [12]. Let us consider the discrete-time N-periodic system described by

$$\begin{cases} x_{k+1} = \displaystyle\sum_{j=0}^{\lceil k \rceil_N} (A_{k,j} x_{k-j} + B_{k,j} w_{k-j}), \\ z_k \;\; = \displaystyle\sum_{j=0}^{\lceil k \rceil_N} (C_{k,j} x_{k-j} + D_{k,j} w_{k-j}). \end{cases} \tag{1}$$

Here, $x_k \in \mathbf{R}^n$, $w_k \in \mathbf{R}^{m_k}$ and $z_k \in \mathbf{R}^{l_k}$. For all $k \geq 0$ and $j \geq 0$, the matrices $A_{k,j} \in \mathbf{R}^{n \times n}$, $B_{k,j} \in \mathbf{R}^{n \times m_{k-j}}$, $C_{k,j} \in \mathbf{R}^{l_k \times n}$, and $D_{k,j} \in \mathbf{R}^{l_k \times m_{k-j}}$ are N-periodic, i.e., $A_{k+N,j} = A_{k,j}$ for all $k \geq 0$, etc. Note that we allow the size of $B_{k,j}$, $C_{k,j}$ and $D_{k,j}$ to be N-periodically time-varying. For the ease of notation, we define

$$M_N := \sum_{k=0}^{N-1} m_k, \quad L_N := \sum_{k=0}^{N-1} l_k.$$

Contrary to the standard state-space description of periodic systems, those matrices $A_{k,j}$, $B_{k,j}$, $C_{k,j}$ and $D_{k,j}$ with $j \neq 0$ in (1) are non-zero. These add to the dynamics delayed effects of some previous states and inputs, only of those located in the same period: $j \in [1, \lceil k \rceil_N]$.

The analysis of the particularly structured system (1) forms an important basis for the PTVDSFC synthesis. To review the analysis results in [11], [12] in a simplified fashion, let us consider the 2-periodic case. Then the equations (1) become

$$\begin{cases} x_{k+1} &= A_{0,0} x_k &+ B_{0,0} w_k, \\ z_k &= C_{0,0} x_k &+ D_{0,0} w_k \end{cases}$$

when k is even and

$$\begin{cases} x_{k+1} = A_{1,0} x_k + A_{1,1} x_{k-1} + B_{1,0} w_k + B_{1,1} w_{k-1}, \\ z_k \;\; = C_{1,0} x_k + C_{1,1} x_{k-1} + D_{1,0} w_k + D_{1,1} w_{k-1} \end{cases}$$

when k is odd. For the performance analysis of this 2-periodic system, we can readily apply the discrete-time system lifting [6] so that we can obtain an equivalent LTI representation of the form

$$\begin{cases} \hat{x}_{k+1} &= \widehat{A}_2 \hat{x}_k &+ \widehat{B}_2 \hat{w}_k, \\ \hat{z}_k &= \widehat{C}_2 \hat{x}_k &+ \widehat{D}_2 \hat{w}_k \end{cases}, \tag{2}$$

$$\begin{bmatrix} \widehat{A}_2 & \widehat{B}_2 \\ \widehat{C}_2 & \widehat{D}_2 \end{bmatrix} := \left[\begin{array}{c|cc} A_{1,0}A_{0,0} + A_{1,1} & A_{1,0}B_{0,0} + B_{1,1} & B_{1,0} \\ \hline C_{0,0} & D_{0,0} & \mathbf{0} \\ C_{1,0}A_{0,0} + C_{1,1} & C_{1,0}B_{0,0} + D_{1,1} & D_{1,0} \end{array}\right].$$

Here, $\hat{x}_k \in \mathbf{R}^n$ while $\hat{w}_k \in \mathbf{R}^{M_2}$, $\hat{z}_k \in \mathbf{R}^{L_2}$. It follows that we can assess the performance of the periodic system (1) by investigating this LTI system.

Similar results readily follow for the cases $N > 2$. Namely, it is always possible to derive an equivalent LTI representation of the form

$$\begin{cases} \hat{x}_{k+1} &= \widehat{A}_N \hat{x}_k &+ \widehat{B}_N \hat{w}_k, \\ \hat{z}_k &= \widehat{C}_N \hat{x}_k &+ \widehat{D}_N \hat{w}_k \end{cases} \tag{3}$$

where $\hat{x}_k \in \mathbf{R}^n$, $\hat{w}_k \in \mathbf{R}^{M_N}$ and $\hat{z}_k \in \mathbf{R}^{L_N}$. We denote the transfer matrix of this LTI system by $T_{N,\hat{z}\hat{w}}(z)$. Then, from [5], we can easily confirm that the system (1) is stable if and only if $\widehat{A}_N$ in (3) is Schur stable. In addition, we can naturally define the H_∞-norm of (1) by $||T_{N,\hat{z}\hat{w}}||_\infty$, which is exactly the same as the H_∞-norm of the LTI system (3). In the time-domain, this norm can be interpreted as an input-to-output l_2 induced norm.

From these observations, it should be fairly easy to derive LMIs that characterize the performance of the system (1). Indeed, all we have to do is simply writing down $\{\widehat{A}_N, \widehat{B}_N, \widehat{C}_N, \widehat{D}_N\}$ in (3) using $A_{k,j}$ ($k = 0, \cdots, N-1, j = 0, \ldots, k$), etc., and applying LMI results for LTI systems. In [11], [12], however, we sought another direction and derived LMI conditions that preserves the matrix structure of the system (1). To review these results, we define $\mathcal{A}_N \in \mathbf{R}^{(N+1)n \times Nn}$, $\mathcal{B}_N \in \mathbf{R}^{(N+1)n \times M_N}$, $\mathcal{C}_N \in \mathbf{R}^{L_N \times Nn}$ and $\mathcal{D}_N \in \mathbf{R}^{L_N \times M_N}$ by

$$\mathcal{A}_N := \begin{bmatrix} A_{N-1,0} & A_{N-1,1} & \cdots & \cdots & A_{N-1,N-1} \\ -\mathbf{1} & A_{N-2,0} & A_{N-2,1} & \cdots & A_{N-2,N-2} \\ \mathbf{0} & -\mathbf{1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & -\mathbf{1} & A_{0,0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & -\mathbf{1} \end{bmatrix},$$

$$\mathcal{B}_N := \begin{bmatrix} B_{N-1,0} & B_{N-1,1} & \cdots & \cdots & B_{N-1,N-1} \\ \mathbf{0} & B_{N-2,0} & B_{N-2,1} & \cdots & B_{N-2,N-2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & B_{0,0} \\ \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \end{bmatrix},$$

$$\mathcal{C}_N := \begin{bmatrix} C_{N-1,0} & C_{N-1,1} & \cdots & \cdots & C_{N-1,N-1} \\ \mathbf{0} & C_{N-2,0} & C_{N-2,1} & \cdots & C_{N-2,N-2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & C_{0,0} \end{bmatrix},$$

$$\mathcal{D}_N := \begin{bmatrix} D_{N-1,0} & D_{N-1,1} & \cdots & \cdots & D_{N-1,N-1} \\ \mathbf{0} & D_{N-2,0} & D_{N-2,1} & \cdots & D_{N-2,N-2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & D_{0,0} \end{bmatrix}.$$

Under these notations, we can state the next results.

Lemma 1 (Stability) [11] The N-periodic system (1) is stable if and only if there exist $X \in \mathbf{P}_n$ and $\mathcal{F} \in \mathbf{R}^{Nn \times (N+1)n}$ such that

$$\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X \end{bmatrix} + \mathrm{He}\{\mathcal{A}_N \mathcal{F}\} \prec \mathbf{0}. \tag{4}$$

Lemma 2 (H_∞ Performance) [12] Let us denote the H_∞-norm of the N-periodic system (1) by ν_N. Then, $\nu_N^2 < \nu$ holds if and only if there exist $X \in \mathbf{P}_n$ and $\mathcal{F}_\infty \in \mathbf{R}^{Nn\times((N+1)n+L_N)}$ such that

$$\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu\mathbf{1}_{L_N} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_N \\ \mathcal{D}_N \end{bmatrix} \begin{bmatrix} \mathcal{B}_N \\ \mathcal{D}_N \end{bmatrix}^T + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A}_N \\ \mathcal{C}_N \end{bmatrix} \mathcal{F}_\infty \right\} \prec \mathbf{0}. \tag{5}$$

In contrast to the fact that the coefficient matrices $\{\widehat{A}_N, \widehat{B}_N, \widehat{C}_N, \widehat{D}_N\}$ of the equivalent LTI system (3) involves $A_{k,j}, B_{k,j}, C_{k,j}$ and $D_{k,j}$ $(k = 0, \cdots, N-1, j = 0, \ldots, k)$ in a very complicated fashion, we see that $\{\mathcal{A}_N, \mathcal{B}_N, \mathcal{C}_N, \mathcal{D}_N\}$ in the LMIs (4) and (5) are affine with respect to $A_{k,j}$, etc. More notably, the LMIs (4) and (5) are *convex with respect to all of the coefficient matrices* $A_{k,j}$, etc. As we clarified in [11], [12] and discuss in more detail in the latter part of the present paper, this crucial property brings great advantage particularly in the case where the system (1) is subject to uncertainties.

2.2 Novel LMI Conditions

We can confirm that, by letting $N = 1$, the LMIs (4) and (5) reduce to the well-known extended LMIs for LTI system analysis [9], [19]. In the LTI case, the results around extended LMIs are more fruitful, and one of the striking achievements is the extended-LMI-based controller synthesis [8], [10]. The key issue has been how we restrict the structure of the additional variables $\mathcal{F}$ or $\mathcal{F}_\infty$ so that we can make these LMIs applicable to controller synthesis.

In the next two lemmas, we show that we can indeed restrict the structure of the additional variables while preserving the *necessity* of the resulting LMIs, even in the case of particularly structured periodic systems. This has been done by carefully reexamining the LMIs in [11], [12] that are merely sufficient (or at least, we cannot proof its necessity) and clearly distinguish the present work from our preceding ones. The proofs of these two lemmas are omitted due to limited space.

Lemma 3 (Stability) The N-periodic system (1) is stable if and only if there exist $X \in \mathbf{P}_n$ and $G_{k,j} \in \mathbf{R}^{n\times n}$ $(k = 0, \cdots, N-1, j = 0, \cdots, k)$ such that

$$\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X \end{bmatrix} + \mathrm{He}\left\{\mathcal{A}_N [\ \mathbf{0}_{Nn,n}\ \mathcal{G}_N\ ]\right\} \prec \mathbf{0}, \tag{6}$$

$$\mathcal{G}_N := \begin{bmatrix} G_{N-1,0} & \cdots & \cdots & G_{N-1,N-1} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & G_{00} \end{bmatrix} \in \mathbf{R}^{Nn\times Nn}. \tag{7}$$

If the LMI (6) is feasible, then $\mathcal{G}_N$ is non-singular.

Lemma 4 (H_∞ Performance) Suppose $B_{k,j} = 0$, $D_{k,j} = 0$ $(j \neq 0)$ in the N-periodic system (1) and let us denote its H_∞-norm by ν_N. Then, $\nu_N^2 < \nu$ holds if and only if there exist $X \in \mathbf{P}_n$ and $G_{k,j} \in \mathbf{R}^{n\times n}$ $(k = 0, \cdots, N-1, j = 0, \cdots, k)$ such that

$$\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu\mathbf{1}_{L_N} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_N \\ \mathcal{D}_N \end{bmatrix} \begin{bmatrix} \mathcal{B}_N \\ \mathcal{D}_N \end{bmatrix}^T + \mathrm{He}\left\{ \begin{bmatrix} \mathcal{A}_N \\ \mathcal{C}_N \end{bmatrix} [\,\mathbf{0}_{Nn,n}\ \mathcal{G}_N\ \mathbf{0}_{Nn,L_N}\,] \right\} \prec \mathbf{0}. \tag{8}$$

Here, $\mathcal{G}_N$ is given in (7). If the LMI (8) is feasible, then $\mathcal{G}_N$ is non-singular.

We have two important remarks regarding Lemmas 3 and 4.

Remark 1 In our preceding work [11], [12], we derived LMI conditions similar to (6) and (8) where we restricted $\mathcal{G}_N$ in (7) to be *block-diagonal*. In a sense this restriction is reasonable since it could be regarded as a natural extension of the LTI-case results [8], [10]. However, if we restrict $\mathcal{G}_N$ to be block-diagonal, the LMI conditions degenerate into merely sufficient (or at least, we cannot prove its necessity). Lemmas 3 and 4 imply that we can obtain necessary and sufficient LMI conditions by letting $\mathcal{G}_N$ to be *block-upper-triangular*, and we conjecture that this fact is strongly related to causality issues. This topic is currently under investigation.

Remark 2 By means of Lemmas 3 and 4 and in particular by exploiting the block-upper-triangular structure of $\mathcal{G}_N$, we can derive LMI conditions for PTVDSFC synthesis as explicated in the next section. It is worth mentioning that Lemma 4 holds under the assumption $B_{k,j} = 0,\ D_{k,j} = 0$ $(j \neq 0)$ in (1), which may seem restrictive. However, as we see in the next section, this assumption is naturally satisfied when we consider practical PTVDSFC synthesis problems.

3 PTVDSFC Synthesis

We are now in a right position to discuss PTVDSFC (Periodically Time-Varying Dynamical State-Feedback Controller) synthesis. To this end, we firstly revisit the structure of what we call PTVDSFCs.

Let us consider the "standard" N-periodic system described by

$$\left\{ \begin{array}{rcl} x_{k+1} & = & A_k x_k + B_k w_k + E_k u_k, \\ z_k & = & C_k x_k + D_k w_k + F_k u_k \end{array} \right. \tag{9}$$

where $x_k \in \mathbf{R}^n$, $u_k \in \mathbf{R}^p$ while $w_k \in \mathbf{R}^{m_k}$ $z_k \in \mathbf{R}^{l_k}$. For this periodic system, we design an N-periodic state-feedback controller of the form

$$u_k = \sum_{j=0}^{[k]_N} K_{k,j} x_{k-j}, \quad K_{k+N,j} = K_{k,j}\ (\forall k \geq 0). \tag{10}$$

This controller is obviously causal, and surely dynamical with (hidden) states of dimension $(N-1)n$. We call this controller N-PTVDSFC. If we let $K_{k,0} = K_k$ and $K_{k,j} = 0$ $(j \neq 0)$, it is obvious that (10) reduces to the standard N-periodic *static* controller

$$u_k = K_k x_k, \quad K_{k+N} = K_k\ (\forall k \geq 0). \tag{11}$$

From (9) and (10), the closed-loop system is described by

$$\begin{cases} x_{k+1} &= \displaystyle\sum_{j=0}^{\lceil k \rceil_N} (A_{k,j}^{\rm cl} x_{k-j} \ + \ B_{k,j}^{\rm cl} w_{k-j}), \\ z_k &= \displaystyle\sum_{j=0}^{\lceil k \rceil_N} (C_{k,j}^{\rm cl} x_{k-j} \ + \ D_{k,j}^{\rm cl} w_{k-j}), \end{cases} \tag{12}$$

where

$$\begin{array}{l} A_{k,0}^{\rm cl} := A_k + E_k K_{k,0}, \ A_{k,j}^{\rm cl} := E_k K_{k,j} \ (j \neq 0) \\ B_{k,0}^{\rm cl} := B_k, \ B_{k,j}^{\rm cl} := \mathbf{0} \ (j \neq 0), \\ C_{k,0}^{\rm cl} := C_k + F_k K_{k,0}, \ C_{k,j}^{\rm cl} := F_k K_{k,j} \ (j \neq 0) \\ D_{k,0}^{\rm cl} := D_k, \ D_{k,j}^{\rm cl} := \mathbf{0} \ (j \neq 0). \end{array} \tag{13}$$

It is apparent that this closed-loop system has exactly the same structure as (1). Moreover, the assumption required in Lemma 4 is naturally satisfied. Thus, if we define by $\mathcal{A}_N^{\rm cl}$ the matrix resulting from $\mathcal{A}_N$ with $A_{k,j}$ replaced by $A_{k,j}^{\rm cl}$ in (13) and define $\mathcal{B}_N^{\rm cl}$, $\mathcal{C}_N^{\rm cl}$ and $\mathcal{D}_N^{\rm cl}$ similarly, we can evaluate the stability and the H_∞ performance of the closed-loop system by simply replacing $\mathcal{A}_N$ by $\mathcal{A}_N^{\rm cl}$, etc., in (6) and (8).

Our goal now is to derive LMI conditions for PTVDSFC synthesis. To this end, we first represent $\mathcal{A}_N^{\rm cl}$ and $\mathcal{C}_N^{\rm cl}$ compactly with respect to the controller parameters $K_{k,j}$ $(k = 0, \cdots, N, \ j = 0, \cdots, k)$. This can be done if we assemble those controller parameters and construct an upper-triangular matrix

$$\mathcal{K}_N := \begin{bmatrix} K_{N-1,0} & \cdots & \cdots & K_{N-1,N-1} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & K_{00} \end{bmatrix} \in \mathbf{R}^{Np \times Nn}. \tag{14}$$

Then, it is straightforward to see that

$$\mathcal{A}_N^{\rm cl} = \mathcal{A}_N^{\rm op} + \mathcal{E}_N^{\rm op} \mathcal{K}_N, \quad \mathcal{C}_N^{\rm cl} = \mathcal{C}_N^{\rm op} + \mathcal{F}_N^{\rm op} \mathcal{K}_N \tag{15}$$

where

$$\mathcal{E}_N^{\rm op} := \begin{bmatrix} E_{N-1,0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \vdots & & \ddots & E_{0,0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} \end{bmatrix} \in \mathbf{R}^{(N+1)n \times Np},$$

$$\mathcal{F}_N^{\rm op} := \begin{bmatrix} F_{N-1,0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & F_{0,0} \end{bmatrix} \in \mathbf{R}^{L_N \times Np}.$$

The definition of $\mathcal{A}_N^{\rm op}$ and $\mathcal{C}_N^{\rm op}$ should be obvious; we define by $\mathcal{A}_N^{\rm op}$ the matrix resulting from $\mathcal{A}_N$ with $A_{k,0}$ replaced by A_k in (9) and $A_{k,j} = \mathbf{0}$ $(j \neq 0)$. The definition of $\mathcal{C}_N^{\rm op}$ follows similarly.

We next consider the *linearizing change of variable*. If we replace $\{\mathcal{A}_N, \mathcal{B}_N, \mathcal{C}_N, \mathcal{D}_N\}$ in the LMIs (6) and (8) by $\{\mathcal{A}_N^{\rm cl}, \mathcal{B}_N^{\rm cl}, \mathcal{C}_N^{\rm cl}, \mathcal{D}_N^{\rm cl}\}$, we encounter a bilinear term among the decision variables $\mathcal{K}_N$ and $\mathcal{G}_N$. However, thanks to the upper-triangular structure of both $\mathcal{K}_N$ and $\mathcal{G}_N$, we can linearize the bilinear term $\mathcal{K}_N \mathcal{G}_N$ via change of variable of the form

$$\mathcal{K}_N \mathcal{G}_N = \mathcal{Y}_N,$$

$$\mathcal{Y}_N = \begin{bmatrix} Y_{N-1,0} & \cdots & \cdots & Y_{N-1,N-1} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & Y_{00} \end{bmatrix} \in \mathbf{R}^{Np \times Nn}. \tag{16}$$

In this way, we can derive the next two theorems that provide LMIs for PTVDSFC synthesis.

Theorem 1 (Stabilizing PTVDSFC Synthesis) Let us consider the N-periodic plant described by (9). Then, there exists a stabilizing N-PTVDSFC of the form (10) if and only if there exist $X \in \mathbf{P}_n$, $\mathcal{G}_N \in \mathbf{R}^{Nn \times Nn}$ and $\mathcal{Y}_N \in \mathbf{R}^{Np \times Nn}$ given respectively in (7) and (16) such that

$$\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X \end{bmatrix} + {\rm He}\{(\mathcal{A}_N^{\rm op} \mathcal{G}_N + \mathcal{E}_N^{\rm op} \mathcal{Y}_N)[\ \mathbf{0}_{Nn,n} \ \mathbf{1}_{Nn}\ ]\} \prec \mathbf{0}. \tag{17}$$

If this LMI is feasible, the desired stabilizing N-PTVDSFC can be constructed via $\mathcal{K}_N = \mathcal{Y}_N \mathcal{G}_N^{-1}$.

Theorem 2 (H_∞-PTVDSFC Synthesis) Let us consider the N-periodic plant described by (9). Then, there exists an N-PTVDSFC of the form (10) that renders the closed-loop system stable and its squared H_∞-norm less than ν if and only if there exist $X \in \mathbf{P}_n$, $\mathcal{G}_N \in \mathbf{R}^{Nn \times Nn}$ and $\mathcal{Y}_N \in \mathbf{R}^{Np \times Nn}$ given respectively in (7) and (16) such that

$$\begin{aligned} &\begin{bmatrix} -X & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu \mathbf{1}_{L_N} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_N^{\rm cl} \\ \mathcal{D}_N^{\rm cl} \end{bmatrix} \begin{bmatrix} \mathcal{B}_N^{\rm cl} \\ \mathcal{D}_N^{\rm cl} \end{bmatrix}^T \\ &+ {\rm He}\left\{ \left(\begin{bmatrix} \mathcal{A}_N^{\rm op} \\ \mathcal{C}_N^{\rm op} \end{bmatrix} \mathcal{G}_N + \begin{bmatrix} \mathcal{E}_N^{\rm op} \\ \mathcal{F}_N^{\rm op} \end{bmatrix} \mathcal{Y}_N \right) \right. \\ &\qquad \left. \times [\ \mathbf{0}_{Nn,n} \ \ \mathbf{1}_{Nn} \ \ \mathbf{0}_{Nn,L_N}\ ] \right\} \prec \mathbf{0}. \end{aligned} \tag{18}$$

If this LMI is feasible, the desired N-PTVDSFC can be constructed via $\mathcal{K}_N = \mathcal{Y}_N \mathcal{G}_N^{-1}$.

We note that, as clearly shown in (14) and (15), our synthesis problem here is nothing but a *structured controller synthesis problem* for which closed-form LMI formulation is not known in general. However, Theorems 1 and 2 shows that this is indeed possible in the case of PTVDSFC synthesis. This is, again, due to the fact that we can restrict $\mathcal{G}_N$ to be block-upper-triangular without introducing any conservatism.

Remark 3 Similarly to Lemmas 1 and 2, in [11], [12], we derived an LMI condition for the H_2 performance analysis of the system (1). Using this LMI and restricting the structure of the additional variables to be upper-triangular as in Lemmas 3 and 4, we can easily derive an LMI condition for the H_2-PTVDSFC synthesis as in Theorems 1 and 2. We expect that this LMI is also necessary and sufficient, but its necessity proof is currently beyond our reach.

4 Robust PTVDSFC Synthesis and Exactness Verification

We believe that the PTVDSFC synthesis based on Theorems 1 and 2 would be effective for a variety of control problems for which definitive approaches are not yet established. As an example, in this section, we consider robust PTVDSFC synthesis.

4.1 Robust PTVDSFC Synthesis

Let us consider the case where the system (9) is subject to a polytopic uncertainty of the form

$$\mathcal{M}_k := \begin{bmatrix} A_k & B_k & E_k \\ C_k & D_k & F_k \end{bmatrix},$$

$$\mathcal{M}_k^{[l]} = \begin{bmatrix} A_k^{[l]} & B_k^{[l]} & E_k^{[l]} \\ C_k^{[l]} & D_k^{[l]} & F_k^{[l]} \end{bmatrix} \ (l = 1, \cdots, L),$$

$$\begin{bmatrix} \mathcal{M}_0 \\ \vdots \\ \mathcal{M}_{N-1} \end{bmatrix} \in \left\{ \sum_{l=1}^{L} \alpha_l \begin{bmatrix} \mathcal{M}_0^{[l]} \\ \vdots \\ \mathcal{M}_{N-1}^{[l]} \end{bmatrix} : \ \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_L \end{bmatrix} \in \boldsymbol{\alpha} \right\}, \tag{19}$$

$$\boldsymbol{\alpha} := \left\{ \alpha : \ \alpha \in \mathbf{R}^L, \ \sum_{l=1}^{L} \alpha_l = 1, \ \alpha_l \geq 0 \right\}.$$

Here, $\mathcal{M}_k^{[l]}$ $(k = 0, \cdots, N-1, \ l = 1, \cdots, L)$ are given matrices that define the vertices of the polytope. For this uncertain plant, our goal here is to design a robust PTVDSFC of the form (10) that minimizes the worst case H_∞-norm of the closed-loop system.

Since basic ideas for robust PTVDSFC are already given in [12], we only state here their outline. The key observation is that the LMI (18) is convex with respect to all of the coefficient matrices A_k $(k = 0, \cdots, N-1)$, etc. In view of this fact, it is obvious that we can design a robust H_∞-PTVDSFC by solving the following SDP:

$$\begin{aligned} \nu_{N,\mathrm{rob}}^2 := \inf_{\mathcal{G}_N, \mathcal{Y}_N, X^{[l]}} \nu \quad & \text{subject to} \\ \begin{bmatrix} -X^{[l]} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-1)n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & X^{[l]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nu \mathbf{1}_{L_N} \end{bmatrix} & + \begin{bmatrix} \mathcal{B}_N^{\mathrm{cl}[l]} \\ \mathcal{D}_N^{\mathrm{cl}[l]} \end{bmatrix} \begin{bmatrix} \mathcal{B}_N^{\mathrm{cl}[l]} \\ \mathcal{D}_N^{\mathrm{cl}[l]} \end{bmatrix}^T \\ + \mathrm{He} \left\{ \left(\begin{bmatrix} \mathcal{A}_N^{\mathrm{op}[l]} \\ \mathcal{C}_N^{\mathrm{op}[l]} \end{bmatrix} \mathcal{G}_N + \begin{bmatrix} \mathcal{E}_N^{\mathrm{op}[l]} \\ \mathcal{F}_N^{\mathrm{op}[l]} \end{bmatrix} \mathcal{Y}_N \right) \right. & \\ \left. \times \left[\begin{array}{ccc} \mathbf{0}_{Nn,n} & \mathbf{1}_{Nn} & \mathbf{0}_{Nn,L_N} \end{array} \right] \right\} \prec \mathbf{0}, & \\ (l = 1, \cdots, L). & \end{aligned} \tag{20}$$

Here, those matrices $\mathcal{A}_N^{\mathrm{op}[l]}$, $\mathcal{B}_N^{\mathrm{cl}[l]}$, $\mathcal{C}_N^{\mathrm{op}[l]}$, $\mathcal{D}_N^{\mathrm{cl}[l]}$, $\mathcal{E}_N^{\mathrm{op}[l]}$, $\mathcal{F}_N^{\mathrm{op}[l]}$ are readily defined from $\mathcal{A}_N^{\mathrm{op}}$, $\mathcal{B}_N^{\mathrm{cl}}$, $\mathcal{C}_N^{\mathrm{op}}$, $\mathcal{D}_N^{\mathrm{cl}}$, $\mathcal{E}_N^{\mathrm{op}}$, $\mathcal{F}_N^{\mathrm{op}}$, respectively, by simply replacing A_k by $A_k^{[l]}$, etc. If the LMIs in (20) hold, the desired robust PTVDSFC can be obtained by $\mathcal{K}_N = \mathcal{Y}_N \mathcal{G}_N^{-1}$.

By using the extended LMI for LTI systems [10] in conjunction with the periodic Lyapunov Lemma [4], [5], we can easily design robust periodically time-varying *static* controller of the form (11). It can be seen that this approach is equivalent to solving the SDP (20) with both $\mathcal{G}_N$ and $\mathcal{Y}_N$ restricted to be block-diagonal. It is also true that our approach in [12] corresponds to solving the SDP (20) with block-diagonal $\mathcal{G}_N$. It follows that the SDP (20) encompasses all of these preceding results. We note that, to obtain a tractable LMI condition, we have enforced a common $\mathcal{G}_N$ for all vertices of the polytope in (20). This is a source of conservatism, and therefore $\nu_{N,\mathrm{rob}}$ still merely gives an upper bound of the achievable H_∞ performance via PTVDSFCs.

4.2 Exactness Verification

In the preceding subsection, we proposed to solve the SDP (20). To refer to the SDP (20) more compactly, let us define by $\alpha^{[l]}$ the l-th vertex of the set $\boldsymbol{\alpha}$ in (19) and denote the left-hand side of the LMI in (20) by $\mathcal{L}_{N\infty}^{\alpha^{[l]}}(\mathcal{G}_N, \mathcal{Y}_N, X^{[l]}, \nu)$. Then, the SDP (20) can be restated concisely as follows:

$$\begin{aligned} \nu_{N,\mathrm{rob}}^2 = \inf_{\mathcal{G}_N, \mathcal{Y}_N, X^{[l]}} \nu \quad & \text{subject to} \\ & \mathcal{L}_{N\infty}^{\alpha^{[l]}}(\mathcal{G}_N, \mathcal{Y}_N, X^{[l]}, \nu) \prec 0 \ (l = 1, \cdots, L). \end{aligned} \tag{21}$$

Extending this notation, we also denote by $\mathcal{L}_{N\infty}^{\alpha}(\mathcal{G}_{N,\alpha}, \mathcal{Y}_{N,\alpha}, X_\alpha, \nu)$ the left-hand of (20) with $\mathcal{A}_N^{\mathrm{op}[l]}$ replaced by $\mathcal{A}_N^{\mathrm{op}}(\alpha) := \sum_{l=1}^{L} \alpha_l \mathcal{A}_N^{\mathrm{op}[l]}$ and $\mathcal{G}_N$ by $\mathcal{G}_{N,\alpha}$, etc., where $\alpha \in \boldsymbol{\alpha}$.

Our goal here is to examine the quality of the upper bound $\nu_{N,\mathrm{rob}}$ in (21) quantitatively. To this end, we make the following definition.

Definition 1 For each fixed $\alpha \in \boldsymbol{\alpha}$, we consider a frozen system in (19) and define by $\nu_N^\star(\alpha)$ the optimal H_∞-norm achieved by the PTVDSFC (10) for this frozen system. Namely, we define

$$\begin{aligned} \nu_N^\star(\alpha)^2 \ := \inf_{\mathcal{G}_{N,\alpha}, \mathcal{Y}_{N,\alpha}, X_\alpha} \nu \quad & \text{subject to} \\ & \mathcal{L}_{N\infty}^{\alpha}(\mathcal{G}_{N,\alpha}, \mathcal{Y}_{N,\alpha}, X_\alpha, \nu) \prec 0. \end{aligned} \tag{22}$$

In addition, we define

$$\nu_N^\star := \max_{\alpha \in \boldsymbol{\alpha}} \nu_N^\star(\alpha). \tag{23}$$

From this definition, we see that the following relation always holds:

$$\nu_{N,\mathrm{rob}} \geq \nu_N^\star. \tag{24}$$

Thus the value $\nu_N^\star$ clearly gives a lower bound of the achievable robust H_∞ performance via PTVDSFC (10). It follows that if we verify that

$$\nu_{N,\mathrm{rob}} = \nu_N^\star \tag{25}$$

holds, we can readily conclude that the designed PTVDSFC is *exact* in the sense that it attains the best achievable performance.

Unfortunately, we cannot expect that (25) always holds. Therefore it is of prime importance to establish a viable test for verifying the exactness. This can be done by relying on the dual of the SDP (21) as summarized in the next theorem.

Theorem 3 Suppose the SDP (21) is feasible and let us denote by $\mathcal{H}_{N,l}$ the resulting dual variable with respect to the l-th LMI in (21). For these $\mathcal{H}_{N,l}$ $(l = 1, \cdots, L)$, suppose there exists $\alpha_{\mathrm{w}} = [\alpha_{\mathrm{w},1}, \cdots, \alpha_{\mathrm{w},L}]^T \in \boldsymbol{\alpha}$ such that

$$\mathcal{H}_{N,l} = \alpha_{\mathrm{w},l} \overline{\mathcal{H}}_N \ (l = 1, \cdots, L), \ \overline{\mathcal{H}}_N := \sum_{l=1}^{L} \mathcal{H}_{N,l}. \tag{26}$$

Then, we can conclude that

$$\nu_{N,\mathrm{rob}} = \nu_N^\star = \nu_N^\star(\alpha_{\mathrm{w}}). \tag{27}$$

Namely, the designed PTVDSFC is exact, and α_{w} corresponds to the worst-case uncertain parameter.

Proof: We first note that, via convex duality theory, the value $\nu_{N,\mathrm{rob}}$ can be characterized as follows:

$$\begin{aligned}\nu_{N,\mathrm{rob}}^2 = \sup_{\mathcal{H}_{N,l}\succeq 0} \ \nu \quad & \text{subject to} \\ & \mathrm{trace}(\sum_{l=1}^{L}\mathcal{H}_{N,l}) = 1 \quad \text{and} \\ & \sum_{l=l}^{L}\mathrm{trace}(\mathcal{L}_{N\infty}^{\alpha^{[l]}}(\mathcal{G}_N,\mathcal{Y}_N,X^{[l]},\nu)\mathcal{H}_{N,l}) \geq 0 \\ & \text{for all} \quad \mathcal{G}_N,\mathcal{Y}_N,X^{[l]} \in \mathbf{P}_n.\end{aligned}$$

Since the optimal dual variable $\mathcal{H}_{N,l}$ $(l = 1,\cdots,L)$ satisfies (26) from our underlying assumption, the above relation implies

$$\begin{aligned}\nu_{N,\mathrm{rob}}^2 = \sup_{\overline{\mathcal{H}}_N\succeq 0} \ \nu \quad & \text{subject to} \\ & \mathrm{trace}(\overline{\mathcal{H}}_N) = 1 \quad \text{and} \\ & \sum_{l=l}^{L}\mathrm{trace}(\mathcal{L}_{N\infty}^{\alpha^{[l]}}(\mathcal{G}_N,\mathcal{Y}_N,X^{[l]},\nu)\alpha_{\mathrm{w},l}\overline{\mathcal{H}}_{N,l}) \geq 0 \\ & \text{for all} \quad \mathcal{G}_N,\mathcal{Y}_N,X^{[l]} \in \mathbf{P}_n\end{aligned}$$

If we define $X(\alpha_{\mathrm{w}}) := \sum_{l=1}^{L}\alpha_{\mathrm{w},l}X^{[l]}$, this can be restated equivalently as

$$\begin{aligned}\nu_{N,\mathrm{rob}}^2 = \sup_{\overline{\mathcal{H}}_N\succeq 0} \ \nu \quad & \text{subject to} \\ & \mathrm{trace}(\overline{\mathcal{H}}_N) = 1 \quad \text{and} \\ & \mathrm{trace}(\mathcal{L}_{N\infty}^{\alpha_{\mathrm{w}}}(\mathcal{G}_N,\mathcal{Y}_N,X(\alpha_{\mathrm{w}}),\nu)\overline{\mathcal{H}}_{N,l}) \geq 0 \\ & \text{for all} \quad \mathcal{G}_N,\mathcal{Y}_N,X^{[l]} \in \mathbf{P}_n.\end{aligned}$$

Again, from the convex duality theory, we can confirm that the value in the right-hand side of the above equality is nothing but $\nu_N^\star(\alpha_{\mathrm{w}})^2$ (defined from (22)). Therefore we have $\nu_{N,\mathrm{rob}} = \nu_N^\star(\alpha_{\mathrm{w}})$. If we further remind (23) and (24), we are led to the conclusion

$$\nu_N^\star \leq \nu_{N,\mathrm{rob}} = \nu_N^\star(\alpha_{\mathrm{w}}) \leq \nu_N^\star.$$

This completes the proof. Q.E.D.

We stress that the exactness verification strategy in Theorem 3 is strongly inspired from [21], [22], [13]. In these studies, LMI relaxations for robust LMI problems are carefully examined, leading to exactness verification tests that exploits the structure of the dual LMIs. Similarly to [21], [22], one of the salient features of Theorem 3 is that, even in the case where (26) does not hold exactly, we can estimate the worst-case uncertain parameter by solving the convex optimization problem

$$\alpha_{\mathrm{w},N}^{\mathrm{est}} := \arg\min_{\alpha\in\boldsymbol{\alpha}} \sum_{l=1}^{L}\sigma_{\max}(\mathcal{H}_{N,l} - \alpha_l\overline{\mathcal{H}}_N). \tag{28}$$

Once we have obtained $\alpha_{\mathrm{w},N}^{\mathrm{est}}$, we can compute $\nu_N^\star(\alpha_{\mathrm{w},N}^{\mathrm{est}})$, which corresponds to a lower bound of the achievable performance. Thus we can examine the quality of the upper bound $\nu_{N,\mathrm{rob}}$ in conjunction with the efficiently-computable lower bound $\nu_N^\star(\alpha_{\mathrm{w},N}^{\mathrm{est}})$.

5 Application to LTI System Synthesis

In [11], [12], we showed that the PTVDSFC synthesis is also promising when dealing with robust control problems for uncertain LTI systems. Here we establish a more rigorous treatment by means of the exactness verification test in Section 4.

Let us consider the polytopic-type uncertain LTI system described by

$$\begin{cases} x_{k+1} = Ax_k + Bw_k + Eu_k, \\ z_k \ \ = Cx_k + Dw_k + Fu_k \end{cases} \tag{29}$$

where

$$\begin{bmatrix} A & B & E \\ C & D & F \end{bmatrix} \in \left\{ \sum_{i=1}^{L}\alpha_i \begin{bmatrix} A^{[i]} & B^{[i]} & E^{[i]} \\ C^{[i]} & D^{[i]} & F^{[i]} \end{bmatrix} : \ \alpha \in \boldsymbol{\alpha} \right\}.$$

By artificially regarding this LTI system as N-periodic (i.e, $A_k = A$ $(k = 0,\cdots,N-1)$ and so on in (9)) and solving the SDP (20), we can design robust H_∞-PTVDSFC. In the following, we discuss the property of this synthesis method by noting that the period N is a parameter that can be freely determined by the designer. To evaluate the control performance for different periods N, it is worth mentioning that the lower bound is uniform, i.e.,

$$\nu_1^\star = \nu_N^\star \quad (\forall N \geq 1). \tag{30}$$

This relation readily follows if we review the well-known fact that, for an uncertainty-free LTI system, the static state-feedback control achieves the best H_∞ performance. With this relation in mind, we firstly state the next two nice properties regarding the PTVDSFC synthesis using (20). Theoretical validation of these two assertions can be done by following similar lines to [12].

1. For given N_1 and N_2, suppose N_2 is a multiple of N_1. Then, we have
$$\nu_1^\star \leq \nu_{N_2,\mathrm{rob}} \leq \nu_{N_1,\mathrm{rob}}. \tag{31}$$
Roughly speaking, this fact implies that we can improve the control performance by simply increasing the period N (at the expense of increased computational burden and controller complexity).
2. If we let $N = 1$, the LMIs in (20) reduce to the well-known extended LMIs for static state-feedback robust controller synthesis [10]. In other words, $\nu_{1,\mathrm{rob}}$ corresponds to the robust static-state feedback result in [10]. Since (31) holds, it turns out that we can obtain no more conservative results than [10] for arbitrarily chosen N.

We next consider the exactness verification of the designed PTVDSFCs. To this end, we can readily apply Theorem 3. If the condition (26) is satisfied for some period N, we can readily conclude that the designed PTVDSFC is exact, achieving $\nu_{N,\mathrm{rob}} = \nu_1^\star$. Otherwise, it would be reasonable to repeat the same procedure by increasing N. In this way, we can construct a hierarchy of SDPs together with the exactness verification tests, which may eventually achieves $\nu_{N,\mathrm{rob}} = \nu_1^\star$. We illustrate the effectiveness of our approach via numerical examples in the next section.

6 Illustrative Examples

Let us consider the polytopic-type uncertain LTI system (29) with three vertices where

$$A^{[1]} = \begin{bmatrix} -0.1 & -0.3 & 0 & -0.7 \\ -0.2 & -0.2 & 0.2 & -0.7 \\ -0.8 & 0.1 & 0 & 0.5 \\ 0.6 & 1.0 & 0.4 & 0.7 \end{bmatrix},$$

$$A^{[2]} = \begin{bmatrix} 0.8 & 0.3 & 0 & 0.1 \\ 1.0 & 0.2 & -0.1 & -0.9 \\ 1.8 & 0.1 & 0.4 & -1.1 \\ 0.2 & -0.1 & 0.7 & -0.5 \end{bmatrix},$$

$$A^{[3]} = \begin{bmatrix} 0.4 & -0.9 & -0.1 & -0.2 \\ 0.5 & -0.4 & 0.1 & -0.5 \\ -0.6 & 0.1 & 0.1 & 0.3 \\ -0.3 & 0.9 & 0.5 & -0.7 \end{bmatrix},$$

$$B^{[1]} = B^{[2]} = B^{[3]} = \begin{bmatrix} 0.9 \\ 0.4 \\ -0.6 \\ 0.5 \end{bmatrix},$$

$$C^{[1]} = C^{[2]} = C^{[3]} = \begin{bmatrix} -0.7 & 0.3 & 0.3 & -0.5 \\ 0 & 0.3 & 0.1 & 0.7 \\ -0.1 & -0.1 & -0.1 & 0.2 \end{bmatrix},$$

$$D^{[1]} = D^{[2]} = D^{[3]} = \begin{bmatrix} 0.7 \\ -0.4 \\ -0.5 \end{bmatrix},$$

$$E^{[1]} = \begin{bmatrix} 0 \\ -0.3 \\ -0.1 \\ 0.8 \end{bmatrix}, \quad E^{[2]} = \begin{bmatrix} -0.2 \\ 0.6 \\ 0.3 \\ 0.2 \end{bmatrix}, \quad E^{[3]} = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0.3 \end{bmatrix},$$

$$F^{[1]} = F^{[2]} = F^{[3]} = \begin{bmatrix} -0.8 \\ -0.2 \\ 0.7 \end{bmatrix}.$$

Our goal here is to design a robust PTVDSFC of the form (10) that minimizes the worst case H_∞-norm of the closed-loop system.

Before examining the results in this paper, we first review the achievements in [12] by solving the SDP (20) with $\mathcal{G}_N$ restricted to be block-diagonal †. In Table 1, we showed the computed upper bound $\nu_{N,\mathrm{rob}}$, the required computation time and the lower bound $\nu_N^\star(\alpha_{\mathrm{w},N}^{\mathrm{est}})$ estimated from (28). We also showed $\nu_{N,\mathrm{rob}}^{\mathrm{analysis}}$, which corresponds to a *recomputed* upper bound of the worst-case H_∞-norm of the close-loop system. Roughly speaking, this computation was carried out by solving the (robust version of the) LMI (5). Furthermore, we attached $\star$ to $\nu_{N,\mathrm{rob}}^{\mathrm{analysis}}$ if the computed value is numerically verified to be exact, i.e., it gives the genuine worst-case H_∞-norm of the closed-loop system. This analysis method is motivated from [17], [20] and will be completed in more detail in [14].

From Table 1, we can confirm that there is a big gap between $\nu_{N,\mathrm{rob}}$ and $\nu_N^\star(\alpha_{\mathrm{w},N}^{\mathrm{est}})$ even in the case $N = 7$. Hence, we cannot say anything on the strictness of the design. With this in mind, we next solved (20) with $\mathcal{G}_N$ being upper-triangular form and obtained the results in Table 2. Contrary to Table 1, we see in Table 2 that $\nu_{N,\mathrm{rob}}$ decreases and $\nu_N^\star(\alpha_{\mathrm{w},N}^{\mathrm{est}})$ increases very rapidly, leading to $\nu_{N,\mathrm{rob}} = \nu_N^\star(\alpha_{\mathrm{w},N}^{\mathrm{est}})$ in the case $N = 7$ (see also Fig. 1). It follows that, by designing PTVDSFC of period seven, we have achieved *exact* synthesis (at the expense of the increased computational burden and controller complexity).

†In this paper, all LMI-related computation computation was carried out with SeDuMi [24] and MATLAB, on PC with CPU Pentium IV 3.6 GHz.

Table 1: Results with block-diagonal $\mathcal{G}_N$ [12].

N	$\nu_{N,\mathrm{rob}}$	cpu-time [s]	$\nu_N^\star(\alpha_{\mathrm{w},N}^{\mathrm{est}})$	$\nu_{N,\mathrm{rob}}^{\mathrm{analysis}}$
1	32.89	0.51	2.01	19.62
2	31.67	0.62	2.67	19.78
3	28.17	0.71	3.49	18.52*
4	26.01	0.97	3.67	17.41*
5	24.40	1.53	4.19	16.97*
6	23.51	2.14	4.68	16.67*
7	22.77	3.04	4.79	16.39*

Table 2: Results with upper-triangular $\mathcal{G}_N$.

N	$\nu_{N,\mathrm{rob}}$	cpu-time [s]	$\nu_N^\star(\alpha_{\mathrm{w},N}^{\mathrm{est}})$	$\nu_{N,\mathrm{rob}}^{\mathrm{analysis}}$
1	32.89	0.52	2.01	19.62
2	21.86	0.63	3.85	19.46
3	14.88	0.89	6.05	14.33*
4	13.66	1.70	9.03	13.58*
5	13.36	3.46	12.37	13.36*
6	13.35	6.01	13.23	13.35*
7	13.34	8.42	13.34	13.34*

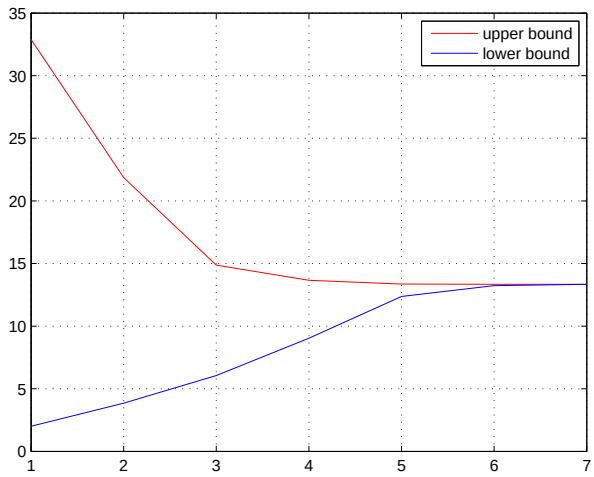

Fig. 1: Computed upper bound $\nu_{N,\mathrm{rob}}$ and lower bound $\nu_N^\star(\alpha_{\mathrm{w},N}^{\mathrm{est}})$ with upper-triangular $\mathcal{G}_N$.

To examine the effectiveness of the PTVDSFC synthesis more carefully, we next extract the worst-case parameter α_{w}. In the case $N = 7$, the exactness test (26) is numerically verified, yielding the worst-case uncertain parameter $\alpha_{\mathrm{w}} = [\ 0\ 1\ 0\ ]^T$. Once we have obtained this parameter, one may think optimistically that the H_∞-optimal static-state feedback controller corresponding to the frozen system with $\alpha = \alpha_{\mathrm{w}}$, say, $\mathcal{K}_1(\alpha_{\mathrm{w}})$, also gives the optimal robust control (achieving the same worst-case H_∞-norm 13.34). In practice, however, the static state-feedback gain

$$\mathcal{K}_1(\alpha_{\mathrm{w}}) = [\ -0.0413 \quad 0.5525 \quad -0.3096 \quad 0.4600\ ]$$

cannot even robustly stabilize the closed-loop system, e.x., we can easily confirm that $A^{[1]} + E^{[1]}\mathcal{K}_1(\alpha_{\mathrm{w}})$ is *not* Schur stable. This fact clearly demonstrates the effectiveness of the PTVDSFC synthesis when we deal with robust control of uncertain LTI systems.

7 Conclusion

In this paper, we derived necessary and sufficient LMI conditions for the existence of the stabilizing and H_∞-PTVDSFCs for discrete-time linear periodic/time-invariant systems. We then applied these LMIs to robust control of polytopic-type uncertain systems, with particular emphasis on the exactness verification of the designed PTVDSFCs. We finally showed via numerical examples that the exact robust control is indeed possible via PTVDSFCs, even for those problems where the standard static state-feedback fails. To summarize, we fully demonstrated the effectiveness of the PTVDSFC synthesis approach when we deal with robust control of uncertain linear discrete-time systems.

Giving more theoretical validation to employing PTVDSFCs, clarifying to what extent we can do with PTVDSFCs in other context of control problems, would be our important future topics.

References

[1] D. Arzelier, D. Peaucelle, C. Farges and J. Daafouz, "Robust Analysis and Synthesis of Linear Polytopic Discrete-Time Periodic Systems via LMIs," Proc. Conference on Decision and Control, pp. 5734–5739 (2005).

[2] V. Balakrishnan and L. Vandenberghe, "Semidefinite Programming Duality and Linear Time-Invariant Systems," *IEEE Transactions on Automatic Control*, Vol. 48, No. 1, pp. 30–41 (2003).

[3] B. R. Barmish, *New Tools for Robustness of Linear Systems*, Macmillan Publishing Company (1994).

[4] S. Bittanti, P. Bolzern and P. Colaneri, "The extended Periodic Lyapunov Lemma," *Automatica*, Volume 21, pp. 603–605 (1985).

[5] S. Bittanti and P. Colaneri, "Analysis of Discrete-Time Linear Periodic Systems," *Control and Dynamic Systems*, Volume 78, pp. 313–339 (1996).

[6] S. Bittanti and P. Colaneri, "Invariant Representations of Discrete-Time Periodic Systems," *Automatica*, Volume 36, pp. 1777–1793 (2000).

[7] S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM (1998).

[8] M. C. de Oliveira, J. Bernussou, and J. C. Geromel. "A New Discrete-time Robust Stability Condition," *Systems & Control Letters*, Volume 37, No. 4, pp. 261–265 (1999).

[9] M. C. de Oliveira and R. E. Skelton, "Stability Tests for Constrained Linear Systems," *Perspectives in Robust Control*, Lecture Notes in Control and Information Sciences 268 edited by S. O. Reza Moheimani, Springer (2001).

[10] M. C. de Oliveira, J. C. Geromel and J. Bernussou. "Extended H_2 and H_∞ Norm Characterizations and Controller Parametrizations for Discrete-Time Systems," *International Journal of Control*, Volume 75, pp. 666–679 (2002).

[11] Y. Ebihara, D. Peaucelle and D. Arzelier, "LMI-based Periodically Time-Varying Dynamical Controller Synthesis for Discrete-Time Uncertain Linear Systems," Proc. the 17th IFAC World Congress, pp. 1354–1359 (2008).

[12] Y. Ebihara, D. Peaucelle and D. Arzelier, "Periodically Time-Varying Dynamical Controller Synthesis for Polytopic-Type Uncertain Discrete-Time Linear Systems," Proc. Conference on Decision and Control, pp. 5438–5443 (2008).

[13] Y. Ebihara, Y. Onishi and T. Hagiwara, "Robust Performance Analysis of Uncertain LTI Systems: Dual LMI Approach and Verifications for Exactness," *IEEE Transactions on Automatic Control*, to appear (2009).

[14] Y. Ebihara, D. Peaucelle and D. Arzelier, "Robustness Analysis of Uncertain Discrete-Time Linear Systems based on System Lifting and LMIs," under preparation (2009).

[15] C. Farges, D. Peaucelle, D. Arzelier and J. Daafouz, "Robust H_2 Performance Analysis and Synthesis of Linear Polytopic Discrete-Time Periodic Systems via LMIs," *Systems and Control Letters*, Vol. 56, pp 159–166 (2007).

[16] L. El Ghaoui and S.-I. Niculescu, *Advances in Linear Matrix Inequality Methods in Control*, SIAM (2000).

[17] T. Hagiwara and Y. Ohara, "Noncausal Linear Periodically Time-Varying Scaling For Discrete-Time Systems," Proc. the 3rd IFAC Workshop on Periodic Control Systems (2007).

[18] P. P. Khargonekar, K. Poolla and A. Tannenbaum, "Robust Control of Linear Time-Invariant Plants Using Periodic Compensation," *IEEE Transactions on Automatic Control*, Volume 30, No. 11, pp. 1088–1096 (1985).

[19] D. Peaucelle, D. Arzelier, O. Bachelier and J. Bernussou, "A New Robust D-Stability Condition for Real Convex Polytopic Uncertainty," *Systems and Control Letters*, Volume 40, pp 21–30 (2000).

[20] D. Peaucelle, Y. Ebihara and D. Arzelier, "Robust H_2 Performance of Discrete-Time Periodic Systems: A New LMI Formulation," Proc. the 17th IFAC World Congress, pp. 1348–1353 (2008).

[21] C. W. Scherer, "Relaxations for Robust Linear Matrix Inequality Problems with Verifications for Exactness," *SIAM Journal on Matrix Analysis and Applications*, Vol. 27, No. 2, pp. 365–395 (2005).

[22] C. W. Scherer, "LMI Relaxations in Robust Control," *European Journal of Control*, Vol. 12, No. 1 (2006).

[23] R. E. Skelton, T. Iwasaki and K. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis (1997).

[24] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB Toolbox for Optimization over Symmetric Cones," *Optimization Methods and Software*, Vol. 11–12, pp. 625–653 (1999).