ROBUST STATIC OUTPUT FEEDBACK STABILIZATION FOR POLYTOPIC UNCERTAIN SYSTEMS: IMPROVING THE GUARANTEED PERFORMANCE BOUND

D. Arzelier ** **D. Peaucelle** ** **S. Salhi** **

** *LAAS-CNRS, 7 Avenue du Colonel Roche, 31 077 Toulouse, Cedex 4, France*

Abstract: A new sufficient condition of robust stabilizability via static output feedback is proposed for polytopic uncertain systems. It is based on a new parameterization of all static output feedback stabilizing gains and uses parameter-dependent Lyapunov functions to systematically reduce conservatism of the usual quadratic stability approach. These results are then extended to deal with the worst-case $\mathcal{H}_2$ guaranteed synthesis problem. A coordinate descent-type algorithm is defined to solve this nonlinear non convex optimization problem. Two numerical examples are provided to illustrate the relevance of the new condition.

Keywords: Output feedback, static controllers, uncertain linear systems, parametric variations, Lyapunov functions.

1. INTRODUCTION

One of the most challenging problem in control theory remains to find numerical tractable necessary and sufficient conditions for the stabilizability of linear time-invariant (LTI) systems via static output feedback. The exact complexity of the problem is not even known, (Blondel *et al.*, 1999). Till now, the most complete solution is the algorithmic approach based on the Tarski-Seidenberg elimination proposed in (Anderson *et al.*, 1975). In the last decade, many different numerical procedures have been defined to deal with this problem, (Iwasaki *et al.*, 1994; Geromel *et al.*, 1996; Grigoriadis and Skelton, 1996; El Ghaoui *et al.*, 1997; Mesbahi, 1998; Peaucelle *et al.*, 2002) to cite a few. (Syrmos *et al.*, 1997) gives a fairly overview of the state of the art at the moment.

In this paper, the model of LTI systems is considered to be not exactly known and some of the defining matrices are supposed to belong to a polytope of matrices. Some attempts have been made to solve this robust synthesis problem (Geromel *et al.*, 1991; Arzelier *et al.*, 1993; de Oliveira *et al.*, 1999; Peaucelle *et al.*, 2000) in the state feedback case. In the static output feedback (SOF) case, the situation is even more complicated and all the references are known to use the so-called quadratic stability framework, (Galimidi and Barmish, 1986; Geromel *et al.*, 1996; Benton and Smith, 1999; Peaucelle and Arzelier, 2001). For such types of problems, a critical tradeoff has to be faced: find testable precise conditions while keeping a weak computational complexity. Robust stability problems have been attacked via methods which rely heavily on the convexity assumption (the results based on quadratic stability concept) or on more complex approaches for which branching operations may be required. In the first case, it is well known that we get too much conservative results while for the second case, computational complexity is a major difficulty.

Due to the nature of the SOF synthesis problem, the proposed conditions are not purely $\mathcal{LMI}$ conditions even in the simplified and conservative

quadratic frameworks. Moreover, it may fail for some instances. Our goal is to provide a new iterative $\mathcal{L}MI$-based method improving in all cases the results given in the quadratic framework. One of the point leading to conservative results in the previous approaches is that the designer imposes to use a single Lyapunov function for the whole uncertainty set since the robust SOF gain is built of this Lyapunov matrix. Using a new parameterization of SOF gains given in (Peaucelle and Arzelier, 2001), a new sufficient condition of robust SOF stabilizability is proposed. This condition involves the search for extra additional variables allowing to decouple the computation of the gain from the one of the parameter-dependent Lyapunov functions. Even if this is not a pure $\mathcal{L}MI$ condition, we show that it is always better than the ones based on quadratic stability. This work is then extended to deal with worst-case $\mathcal{H}_2$ performance synthesis problems showing an important reduction of conservatism in the computation of guaranteed worst-case bound. A cross-decomposition algorithm similar to the one given in (Peaucelle and Arzelier, 2001) may be designed for the performance problem. Two academic examples show the relevance and effectiveness of the new approach.

For conciseness reasons, some abbreviations are used. $co\{A^1, \cdots, A^N\}$ is the convex hull of the polytope defined by its vertices. $sym(A) = A + A'$. $[\star]'BA = A'BA$ and

$$\begin{bmatrix} A & B \\ \star & C \end{bmatrix} = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}$$

2. PROBLEM STATEMENT AND PRELIMINARIES

We will consider the following linear time-invariant continuous-time system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector and $y \in \mathbb{R}^r$ is the vector of measured outputs. Moreover,

$$\begin{aligned} M &= \begin{bmatrix} A & B \end{bmatrix} \in \Omega \\ &= co\left\{ \begin{bmatrix} A^1 & B^1 \end{bmatrix}, \cdots, \begin{bmatrix} A^N & B^N \end{bmatrix} \right\} \end{aligned} \tag{2}$$

Hence, Ω is a convex polytope of matrices for which each element may be expressed as a convex combination of the N vertices of Ω:

$$\begin{aligned} M(\xi) &= \begin{bmatrix} A(\xi) & B(\xi) \end{bmatrix} \\ &= \sum_{i=1}^{N} \xi_i \begin{bmatrix} A^i & B^i \end{bmatrix} \\ &\xi \in \Xi = \left\{ \xi = \sum_{i=1}^{N} \xi_i = 1 \quad \xi_i \geq 0 \right\} \end{aligned} \tag{3}$$

The uncertain vector of parameters ξ is supposed to be time-invariant so that the realization of the model (2) is not known but is not time-varying.

The first problem addressed in this paper is to find a robustly stabilizing static output feedback control law $u(t) = Ky(t)$ for the model (2) i.e. to find a single gain matrix $K \in \mathbb{R}^{r \times m}$ such that every member of the polytope:

$$\Omega_{bf} = co\left\{A^1 + B^1KC, \cdots, A^N + B^NKC\right\} \tag{4}$$

maintains eigenvalue location in the left-half plane.
It is well-known that the test of the stability of a matrix polytope is equivalent to solving an $\mathcal{NP}$-hard problem, (Coxson and Demarco, 1991). Even if the particular complexity of the problem of stabilization via static output feedback is not known, one may conjecture that the related problem of robust stabilization via static output feedback of a polytope of matrices is therefore equivalent to an $\mathcal{NP}$-hard problem. Usually, two different robust stability concepts may been defined to study robust stability of a matrix A belonging in a convex matrix polytope.

Definition 1. :

- Ω is said to be quadratically stabilizable via static output feedback if and only if there exist a positive definite matrix $X \in \mathbb{R}^{n \times n}$ and a matrix $K \in \mathbb{R}^{r \times m}$ such that:

$$[\star]'[\star]' \begin{bmatrix} \mathbf{0} & X \\ X & \mathbf{0} \end{bmatrix} \begin{bmatrix} A(\xi) & B(\xi) \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ KC \end{bmatrix} < \mathbf{0} \tag{5}$$
$$\forall\, \xi \in \Xi$$

- Ω is said to be robustly stabilizable via static output feedback if and only if, for each $M = \begin{bmatrix} A & B \end{bmatrix}$, there exist a positive definite matrix $X \in \mathbb{R}^{n \times n}$ and a matrix $K \in \mathbb{R}^{r \times m}$ such that:

$$[\star]'[\star]' \begin{bmatrix} \mathbf{0} & X(\xi) \\ X(\xi) & \mathbf{0} \end{bmatrix} \begin{bmatrix} A(\xi) & B(\xi) \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ KC \end{bmatrix} < \mathbf{0} \tag{6}$$

Quadratic stability has proven to be useful to derive simple and tractable conditions using only vertex matrices. This approach referred to as vertexization in (Barmish and Shcherbakov, 2000) may be extended to the SOF case. Hence, a straightforward necessary and sufficient condition of quadratic stabilizability via SOF is the following:

Theorem 1. (2) is quadratically stabilizable via SOF if and only if there exist a positive definite matrix $X \in \mathbb{R}^{n \times n}$ and a matrix $K \in \mathbb{R}^{r \times m}$ such that:

$$[\star]'[\star]' \begin{bmatrix} \mathbf{0} & X \\ X & \mathbf{0} \end{bmatrix} \begin{bmatrix} A^i & B^i \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ KC \end{bmatrix} < \mathbf{0} \tag{7}$$
$$\forall\, i = 1, \cdots, N$$

Unfortunately, this condition is useless as it stands. In (Geromel *et al.*, 1996), a necessary and sufficient condition of quadratic stabilizability via SOF is provided in terms of the intersection of a convex set and a set defined by a nonlinear real valued function. In (Benton and Smith, 1999), an $\mathcal{L}MI$-based procedure is proposed for which a particular initialization is required. Moreover, extension of this approach to deal with robust guaranteed performance seems hardly possible. The next section will provide some alternatives to previous works.

3. ROBUST AND QUADRATIC STABILIZATION VIA STATIC OUTPUT FEEDBACK

Some results based on the quadratic framework (Peaucelle and Arzelier, 2001) and complete proofs are recalled. The main purpose of this section is then to propose a new approach encompassing the quadratic one in all cases.

3.1 *Quadratic stabilization via SOF*

Theorem 2. (Peaucelle and Arzelier, 2001) The system (1) is quadratically stabilizable via static output feedback if and only if there exist a positive definite matrix $X \in \mathbb{R}^{n\times n}$, matrices $K_{sf} \in \mathbb{R}^{m\times n}$, $Z \in \mathbb{R}^{m\times r}$, $F \in \mathbb{R}^{m\times m}$ such that:

$$\begin{aligned}&[\star]' \begin{bmatrix} \mathbf{0} & X \\ X & \mathbf{0} \end{bmatrix} \begin{bmatrix} A^i & B^i \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \\ &+sym\left\{ \begin{bmatrix} K_{sf}' \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} ZC & F \end{bmatrix} \right\} < \mathbf{0} \\ &\forall\ i = 1, \cdots, N\end{aligned} \tag{8}$$

A quadratically stabilizing SOF is then given by $K = -F^{-1}Z$.

Lemma 1. (Peaucelle and Arzelier, 2001)

If the condition (8) is verified for a positive definite matrix X and matrices K_{sf}, Z and F of the appropriate dimensions then K_{sf} is necessarily a quadratically stabilizing state feedback matrix for the system (10).

3.2 *Improved conditions via PDLF*

The main advantage of the new parameterization of SOF given in the previous subsection is that it allows to decouple the computation of the SOF gain and the computation of Lyapunov matrix allowing to derive a new sufficient condition of robust SOF stabilizability based on parameter-dependent Lyapunov functions.

Theorem 3. If there exist N positive definite matrices $X^i \in \mathbb{R}^{n\times n}$, matrices $E_1 \in \mathbb{R}^{n\times n}$, $E_2 \in \mathbb{R}^{n\times n}$, $K_{sf} \in \mathbb{R}^{m\times n}$, $Z \in \mathbb{R}^{m\times r}$, $F \in \mathbb{R}^{m\times m}$ solutions, $\forall\ i = 1, \cdots, N$, of:

$$\begin{aligned}&\begin{bmatrix} \mathbf{0} & X^i & \mathbf{0} \\ X^i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + sym\left\{ \begin{bmatrix} E_1 \\ E_2 \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} A^i & -\mathbf{1} & B^i \end{bmatrix} \right\} \\ &+sym\left\{ \begin{bmatrix} K_{sf}' \\ \mathbf{0} \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} ZC & \mathbf{0} & F \end{bmatrix} \right\} < \mathbf{0}\end{aligned} \tag{9}$$

then $K = -F^{-1}Z$ is a robustly stabilizing SOF.

Note that two different sets of extra variables are introduced here. K_{sf}, Z and F are used to define the new parameterization of the SOF gains while E_1 and E_2 allows to involve parameter-dependent Lyapunov functions. An interpretation of the extra variables K_{sf} similar to the one in quadratic setup is now proposed.

Lemma 2. If the condition (9) is verified for$N$ positive definite matrices X^i and matrices K_{sf}, Z and F of the appropriate dimensions then K_{sf} is necessarily a robustly stabilizing state feedback matrix for the system (10).

The key point of the new sufficient condition is that it ensures to always get better results than the quadratic approach. This fact is formalized in the following theorem.

Theorem 4. If (1) is quadratically stabilizable via static output feedback then there exist N positive definite matrices $X^i \in \mathbb{R}^{n\times n}$, matrices $E_1 \in \mathbb{R}^{n\times n}$, $E_2 \in \mathbb{R}^{n\times n}$, $K_{sf} \in \mathbb{R}^{m\times n}$, $Z \in \mathbb{R}^{m\times r}$, $F \in \mathbb{R}^{m\times m}$ solutions, $\forall\ i = 1, \cdots, N$, of (9).

One of the feature of the new condition is that it may be easily extended to deal with the problem of worst-case guaranteed performance. In the next section, a procedure for worst-case guaranteed $\mathcal{H}_2$ stabilization is derived.

4. ROBUST $\mathcal{H}_2$ SUBOPTIMAL STABILIZATION VIA STATIC OUTPUT FEEDBACK

Let the uncertain model of the system be given by:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_2 w(t) + Bu(t) \\ z_2(t) &= C_2 x(t) + D_{2u} u(t) \\ y(t) &= Cx(t)\end{aligned} \tag{10}$$

where $z_2 \in \mathbb{R}^{r_2}$ is the vector of controlled outputs and $w_2 \in \mathbb{R}^{m_2}$ the vector of exogenous disturbances. Moreover, polytopic uncertainty affects the model:

$$\begin{bmatrix} A & B_2 & B \\ C_2 & D_{2u} & \mathbf{0} \end{bmatrix} \in \Omega_2 \tag{11}$$

where:

$$\begin{bmatrix} A & B_2 & B \\ C_2 & D_{2u} & \mathbf{0} \end{bmatrix} = \sum_{i=1}^{N} \xi_i \begin{bmatrix} A^i & B_2^i & B^i \\ C_2^i & D_{2u}^i & \mathbf{0} \end{bmatrix}$$
$$\sum_{i=1}^{N} \xi_i = 1 \quad \xi_i \geq 0 \tag{12}$$

Let us define $\mathcal{K}$ the set of stabilizing SOF gains K and for a given $K \in \mathcal{K}$, the transfer matrix between z_2 and w_2 by $T_{z_2w_2}(s, K, \xi)$. Note that the matrices characterizing the $\mathcal{H}_2$ performance are assumed to be uncertain while no regularity assumptions are assumed. Since, for $K \in \mathcal{K}$, each realization of the model in Ω_2 is strictly proper and asymptotically stable, the $\mathcal{H}_2$ norm of the transfer may be defined. We want to find a robustly stabilizing SOF gain K minimizing the worst-case $\mathcal{H}_2$ norm of the transfer $T_{z_2w_2}(s, K, \xi)$ over Ω_2.

Problem 1. worst-case $\mathcal{H}_2$ SOF stabilization

$$\begin{aligned} ||T_{z_2w_2}(s)||_{2,w.c.} &= \min_{K\in\mathcal{K}} \max_{\xi\in\Xi} ||T_{z_2w_2}(s, K, \xi)||_2 \\ &= \min_{K\in\mathcal{K}} ||T_{z_2w_2}(s, K)||_{2,w.c.} \end{aligned} \tag{13}$$

This problem is a non convex optimization problem very hard to solve. Except by using complex approaches involving branching operations, the best one may expect is to get tractable conditions with associated numerical procedures giving a suboptimal solution to this problem.

Problem 2. w.c. guaranteed $\mathcal{H}_2$ SOF stabilization

$$\begin{aligned} \gamma^* = \min_{K\in\mathcal{K}} \ & \gamma(K) \\ \text{under} \quad & ||T_{z_2w_2}(s)||^2_{2,w.c.} \leq \gamma(K) \end{aligned} \tag{14}$$

Different methods to tackle this problem are now presented.

4.1 *Quadratic $\mathcal{H}_2$ suboptimal stabilization*

First, the quadratic approach proposed in (Peaucelle and Arzelier, 2001) is recalled. In the quadratic context, a different solution based on cutting planes may be considered, (Geromel *et al.*, 1996). In the section devoted to numerical examples, all these methods are compared.

Theorem 5. (Peaucelle and Arzelier, 2001)

If there exist a positive definite matrices $X \in \mathbb{R}^{n\times n}$, a positive definite matrix $T \in \mathbb{R}^{m_2\times m_2}$, matrices $K_{sf} \in \mathbb{R}^{m\times n}$, $Z \in \mathbb{R}^{m\times r}$, $F \in \mathbb{R}^{m\times m}$ and $\gamma_q > 0$ solutions, $\forall\ i = 1, \cdots, N$, of:

$$\begin{aligned} & trace(T) < \gamma_q \\ & \begin{bmatrix} -T & B_{2i}'X \\ XB_{2i} & X \end{bmatrix} < \mathbf{0} \\ & \begin{bmatrix} A_i & B_i' \\ \mathbf{1} & \mathbf{0} \end{bmatrix}' \begin{bmatrix} \mathbf{0} & X \\ X & \mathbf{0} \end{bmatrix} [\star] + \begin{bmatrix} C_{2i}' \\ D_{2ui}' \end{bmatrix} \begin{bmatrix} C_{2i} & D_{2ui} \end{bmatrix}' \\ & +sym\left\{ \begin{bmatrix} K_{sf}' \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} ZC & F \end{bmatrix} \right\} < \mathbf{0} \end{aligned} \tag{15}$$

then $K = -F^{-1}Z$ is a quadratically suboptimal $\mathcal{H}_2$ SOF such that

$$||T_{z_2w_2}(s)||^2_{2,w.c.} \leq \gamma_q \tag{16}$$

Related to this result, an optimization problem may be defined:

$$\begin{aligned} \gamma_q^* = & \min_{\gamma_q, K_{sf}, X, Z, F} \ \gamma_q \\ \text{under} & \quad (15) \end{aligned} \tag{17}$$

The insight into the genuine nature of the variable K_{sf} given in lemma 1 is used in (Peaucelle and Arzelier, 2001) to initialize a cross-decomposition algorithm in order to solve (17). This algorithm is not reminded here since a similar one will be presented in the next subsection.

4.2 *Robust $\mathcal{H}_2$ suboptimal stabilization via PDLF*

The result of subsection 3.2 is now extended to deal with the worst-case guaranteed $\mathcal{H}_2$ SOF stabilization problem.

Theorem 6. If there exist N positive definite matrices $X^i \in \mathbb{R}^{n\times n}$, a positive definite matrix $T \in \mathbb{R}^{m_2\times m_2}$, matrices $G_1 \in \mathbb{R}^{m_2\times n}$, $G_2 \in \mathbb{R}^{n\times n}$, $E_1 \in \mathbb{R}^{n\times n}$, $E_2 \in \mathbb{R}^{n\times n}$, $K_{sf} \in \mathbb{R}^{m\times n}$, $Z \in \mathbb{R}^{m\times r}$, $F \in \mathbb{R}^{m\times m}$ and $\gamma_l > 0$ solutions, $\forall\ i = 1, \cdots, N$, of:

$$\begin{aligned} & trace(T) < \gamma_l \\ & \begin{bmatrix} -T & \mathbf{0} \\ \mathbf{0} & X^i \end{bmatrix} + sym\left\{ \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \begin{bmatrix} B_2^i & -\mathbf{1} \end{bmatrix} \right\} < \mathbf{0} \\ & \begin{bmatrix} \mathbf{0} & X^i & \mathbf{0} & C_2^{i'} \\ X^i & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D_{2u}^{i'} \\ C_2^i & \mathbf{0} & D_{2u}^i & -\mathbf{1} \end{bmatrix} \\ & +sym\left\{ \begin{bmatrix} E_1 \\ E_2 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} A^i & -\mathbf{1} & B^i & \mathbf{0} \end{bmatrix} \right\} \\ & +sym\left\{ \begin{bmatrix} K_{sf}' \\ \mathbf{0} \\ -\mathbf{1} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} ZC & \mathbf{0} & F & \mathbf{0} \end{bmatrix} \right\} < \mathbf{0} \end{aligned} \tag{18}$$

1- **Initialization:** $k = 1$
Solve the following $\mathcal{L}MI$ problem for some $\epsilon > 0$ with respect to the N positive definite matrices X^i and matrices H and G:

$$\begin{bmatrix} \epsilon sym\left\{A^i H + B^i G\right\} & \star \\ X^{i'} - \epsilon H + H' A^i + G' B^{i'} & -H - H' \end{bmatrix} < \mathbf{0} \quad (22)$$

If it has a solution then $K_{sf,1} = GH^{-1}$ is a robust state feedback for $(A, B) \in \Omega$.

2- **Step** k**:**
2.1 Solve the $\mathcal{L}MI$ minimization problem (18) with $K_{sf} = K_{sf,k}$ and with respect to the N positive definite matrices X^i and matrices Z and F.
$\rightarrow \gamma_l^{k,1}$
2.2 Solve the $\mathcal{L}MI$ minimization problem (18) with $Z = Z_k$ and $F = F_k$ and with respect to the N positive definite matrices X^i and matrix K_{sf}.
$\rightarrow \gamma_l^{k,2}$

3- **Termination step:** If $\gamma_l^{k-1,2} - \gamma_l^{k,2} < \eta$ then stop.

$$K_{sub.} = -F_k^{-1} Z_k \quad (23)$$

otherwise $k \leftarrow k + 1$ and go step 2.

then $K = -F^{-1}Z$ is a robust suboptimal $\mathcal{H}_2$ static output feedback such that

$$||T_{z_2 w_2}^{w.c.}||_2^2 \leq \gamma_l \quad (19)$$

In the same way as for the quadratic case, a nonlinear optimization problem may be defined:

$$\begin{array}{lcl} \gamma_l^* = & \min\limits_{\substack{\gamma_q, K_{sf}, X^i, Z, F \\ E_1, E_2, G_1, G_2}} & \gamma_l \\ \text{under} & & (18) \end{array} \quad (20)$$

Following theorem 4, it is easy to prove the following lemma.

Lemma 3. :

$$||T_{z_2 w_2}(s)||_{2,w.c.}^2 \leq \gamma_l^* \leq \gamma_q^* \quad (21)$$

4.3 *A coordinate descent-type algorithm*

The solution of optimization problem (18) involves the solution of bilinear matrix inequality and the reader may think that no advance has been done so far. In fact, the main advantage of the new formulation is that additional degrees of freedom allow now to define an iterative procedure based on $\mathcal{L}MI$ optimization converging to a local optimal solution. The set of extra variables may then be split in two. At each iteration, an $\mathcal{L}MI$ optimization problem is solved with respect to a set of variables while the other is frozen.

The initialization step is particularly interesting since we need to solve a problem of robust stabilization via state feedback of the polytope Ω. However, in some cases, a quadratically stabilizing state feedback may be employed to initialize the problem. Of course, in any case, no guarantee is given that the chosen initial K_{sf} will lead to a feasible second step even if the polytope Ω_2 is robustly stabilizable via SOF. Nevertheless, this procedure provides a sequence of feasible solutions and a nonincreasing sequence of criteria.

Theorem 7. The sequence $\{\gamma_l^{k,i}\}_{k,i}$ is a monotonic nonincreasing sequence converging to a local minimum.

Another interesting point in the previous algorithm is that at each semi iteration 2.1 or 2.2, the complete set of N parameter-dependent Lyapunov functions is included in the set of decision variables and one can hope to shape this set to get better performance bounds.

5. NUMERICAL EXAMPLES

5.1 *Example 1*

The previous algorithm is now applied to the robust stabilization problem of an uncertain continuous-time system defined by the following system matrices:

$$A = \begin{bmatrix} 0 & \alpha - 1 \\ \beta & 0 \end{bmatrix} \quad B = \begin{bmatrix} \alpha \\ 1 - \beta \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad C_2 = \mathbf{1}_2 \qquad D_{2u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The uncertain parameters are defined as $|\alpha - 0.5| \leq \gamma$, $|\beta - 0.5| \leq \gamma$. This defines a polytope of matrices (A, B) with four vertices. This example is borrowed from (Colaneri *et al.*, 1997) where a quadratic state feedback stabilizing gain is computed and consequently modified for static output feedback purposes. Note that for $\gamma = 0.5$, two of the vertices of the polytope are uncontrollable pairs. For a given value of $\gamma \in (0\ ,\ 0.5[$, a necessary and sufficient condition of robust stabilizability via static ouput feedback is

$$K < 0 \text{ and } K > \frac{\gamma - 0.5}{\gamma + 0.5} \quad (24)$$

In (Colaneri *et al.*, 1997), it is shown that the set of quadratic state feedback stabilizing gains is not empty for all $\gamma \in (0\ ,\ 0.36)$. For $\gamma = 0.4$, the pair $(A\ ,\ B)$ is therefore not quadratically stabilizable via static output feedback. Using the algorithm 4.3 with $\epsilon = 10^{-14}$ for the initialization step, we

get a suboptimal static output feedback gain and the actual worst-case $\mathcal{H}_2$ performance:

$$\begin{array}{l} K_{sub.} = -0.0108 \\ ||T_{z_2w_2}(K_{sub.})||_{2w.c.} = 71.17 \end{array} \tag{25}$$

To get an idea of the conservatism of our approach, a brute force method has been used by gridding the space of uncertain parameters and the interval of admissible robust static output feedbacks given by (24).

$$\begin{array}{l} K^* = -0.0572 \\ ||T_{z_2w_2}(K^*)||_{2w.c.} = 41.42 \end{array} \tag{26}$$

This example is particularly interesting since it shows that the new sufficient condition clearly outperforms the previous ones based on quadratic stability and may be a valuable alternative when this last approach fails. Obviously, the result may be rather conservative due to the particular chosen initialization.

5.2 *Example 2*

The proposed exemple is based on the model of the linearized equations of the VTOL helicopter borrowed from (Keel *et al.*, 1988) and modified as in (Geromel *et al.*, 1996). The quadratic approach developed in (Geromel *et al.*, 1996) and in (Peaucelle and Arzelier, 2001) respectively denoted GPS96 and PA01 is then compared to the one proposed in this paper. The algorithm (4.3) gives its result after 10 iterations, (20 $\mathcal{L}MI$ iterations). The initialization K_{sf}^{init} is obtained for $\epsilon = 1$.

$$K_{sub.} = \begin{bmatrix} 1.541 \\ -10.252 \end{bmatrix} \tag{27}$$

	GPS96	PA01	(18)
Guaranteed $\mathcal{H}_2$ bound	4.5	4.66	**3.9651**

Once again, the new method gives a much better result than the previous ones based on quadratic framework.

6. CONCLUSIONS

A new sufficient condition involving a bilinear matrix inequality for the robust stabilization of polytope of matrices via static output feedback has been proposed. This new condition relies on a new parameterization of all stabilizing SOF gains which allows the decoupling between the computation of the SOF gain and the computation of parameter-dependent Lyapunov functions. It is shown that it necessarily outperforms previous conditions based on the use of a single Lyapunov function for the whole set of uncertainty. Extensions to deal with $\mathcal{H}_2$ performance criterion are also presented.

Some extensions of this work are in progress. First, A precise characterization of the set of possible initialization for the cross-decomposition algorithm is the crucial step giving a real insight to the relevance of the numerical procedure. Extentions of this approach to different kinds of performance such as $\mathcal{H}_\infty$ or pole placement may be easily considered. Finally, it remains to understand how this new parameterization may be used for the robust fixed order control design problems.

7. REFERENCES

Anderson, B.D.O., N.K. Bose and E.I. Jury (1975). Output feedback stabilization and related problems - solutions via decision methods. *IEEE Trans. on Automat. Control.*

Arzelier, D., J. Bernussou and G. Garcia (1993). Pole assignment of linear uncertain systems in a sector via a lyapunov-type approach. *IEEE Transactions on Automatic Control* **38**(7), 1128–1132.

Barmish, B.R. and P.S. Shcherbakov (2000). On avoiding vertexization of robustness problems: the approximate feasibility concept. In: *39th Conference on decision and Control.* IEEE.

Benton, R.E. and D. Smith (1999). A non-iterative lmi-based algorithm for robust static-output-feedback stabilization. *Int. J. Control* **72**(14), 1322–1330.

Blondel, V.D., E.D. Sontag, M. Vidyasagar and J.C. Willems (1999). *Open problems in mathematical systems and control theory.* Springer.

Colaneri, P, J.C Geromel and A. Locatelli (1997). *Control Theory and Design : an RH_2 and RH_∞ viewpoint..* Academic Press.

Coxson, G.E. and C.L. Demarco (1991). Testing robust stability of general matrix polytopes is an np-hard computation. In: *Annual Allerton Conference on Communication, Control and Computing.* Monticello, Illinois. pp. 105–106.

de Oliveira, M.C., J.C. Geromel and J. Bernussou (1999). An LMI optimization approach to multiobjective and robust H_∞ controller design for discrete-time systems. In: *Control and Decision Conference.* Phoenix, Ar.

El Ghaoui, L., F. Oustry and M. AitRami (1997). A cone complementarity linearization algorithm for static ouput-feedback and related problems. *IEEE Trans. on Automat. Control* **42**(8), 1171–1176.

Galimidi, A.R. and B.R. Barmish (1986). The constrained lyapunov problem and its application to robust output feedback stabilization. *IEEE Trans. on Automat. Control* **31**(5), 410–419.

Geromel, J.C., P.L.D. Peres and J. Bernussou (1991). On a convex parameter space method for linear control design of uncertain systems. *SIAM J. Control and Optimization* **29**(2), 381–402.

Geromel, J.C., P.L.D. Peres and S.R. Souza (1996). Convex analysis of output feedback control problems: Robust stability and performance. *IEEE Trans. on Automat. Control* **41**(7), 997–1003.

Grigoriadis, K.M. and R.E. Skelton (1996). Low order design for LMI problems using alternating projection methods. *Automatica* **32**(8), 1117–1125.

Iwasaki, T., R.E. Skelton and J.C. Geromel (1994). Linear quadratic suboptimal control with static output feedback. *Systems & Control Letters* **23**(6), 421–430.

Keel, L., S.P. Bhattacharyya and J.W. Howze (1988). Robust control with structured perturbations. *IEEE Trans. on Automat. Control* **33**, 68–78.

Mesbahi, M. (1998). Solving a class of rank minimization problems as semi-definite programs with applications to fixed order output feedback synthesis. *Systems & Control Letters* **33**, 31–36.

Peaucelle, D. and D. Arzelier (2001). An efficient numerical solution for H_2 static output feedback synthesis. In: *European Control Conference*. Porto, Portugal. pp. 3800–3805.

Peaucelle, D., D. Arzelier and R. Bertrand (2002). Ellipsoidal sets for static output feedback. In: *15th IFAC World Congress*. Barcelona.

Peaucelle, D., D. Arzelier, O. Bachelier and J. Bernussou (2000). A new robust D-stability condition for real convex polytopic uncertainty. *Systems & Control Letters* **40**(1), 21–30.

Syrmos, V.L., C.T. Abdallah, P. Dorato and K. Grigoriadis (1997). Static output feedback: A survey. *Automatica* **33**(2), 125–137.