New $\mathcal{LMI}$-based conditions for robust $\mathcal{H}_2$ performance analysis

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Abstract

This paper presents robust $\mathcal{H}_2$ performance analysis results for LTI systems with real parametric uncertainty. New conditions based on parameter-dependent Lyapunov functions are proposed. The results encompass a wide variety of uncertain systems since an $\mathcal{L}(\Sigma, \Delta)$-type modelling of uncertainties is considered. In the particular case when the modelling reduces to affine polytopic uncertainty, three types of parameter-dependent Lyapunov functions are compared. Valuable conclusions are inferred from a numerical example.

1 Introduction

For robust stability analysis purpose, two different types of stability are usually encountered in the literature. While robust stability ensures that the uncertain system is stable for all admissible constant and unknown parameter realisations, quadratic stability guarantees stability via the existence of a quadratic Lyapunov function for all possibly time-varying parameter variations. As a result this approach proves to be very conservative. More sophisticated sufficient conditions for robust stability may be derived in the context of Lyapunov theory by using extra variables known as scaling variables or multipliers.
Another approach to fill up the gap between robust and quadratic stability consists in looking after parameter-dependent Lyapunov functions. It is well-known that considering parameter-dependent Lyapunov functions instead of a fixed quadratic Lyapunov function leads to less conservative robust stability conditions, [?],[?],[?],[?],[?],[?],[?]. In all these references, except [?],[?],[?], parameter-dependent Lyapunov functions which are affine or multiaffine with respect to parameters are proposed. For LFT type uncertainty, this proves to be not sufficient and a more complex dependence may be required. In this paper, it is proposed to search for "quadratic-LFT" parameter-dependent Lyapunov functions, [?]. This form naturally shows up from LFT scaling technique in [?].

Moreover, this "quadratic-LFT" parameter-dependent Lyapunov functions setting, is used for robust performance analysis. We are interested in checking whether the $\mathcal{H}_2$ norm of the uncertain system remains under some level in the presence of uncertainty, leading to the robust $\mathcal{H}_2$ performance problem. Our aim is to develop tractable optimisation problems that converge to such robust guaranteed $\mathcal{H}_2$ costs.
The contribution of this paper is twofold:

- For any LFT-type uncertainty, a new robust stability test based on "quadratic-LFT" parameter-dependent Lyapunov functions has been developed.
- In the particular case of affine polytopic uncertainty, two other new methods are proposed. They are based on linear parameter-dependent Lyapunov functions. Hence, for affine polytopic modelling of uncertainty, three different choices of parameter-dependent Lyapunov matrices are available.

These results are illustrated by a numerical example from [?].

2 Problem statement

Let us consider the following Linear Time-Invariant (LTI), continuous-time, uncertain system:

$$\left\{ \begin{array}{l} \dot{x}(t) = A(\Delta)x(t) + Bw(t) \\ z(t) = C(\Delta)x(t) \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^q$ is the disturbance vector, $z \in \mathbb{R}^p$ is the controlled output vector and the uncertainty Δ is time-invariant and belongs to a given closed convex set $\mathbf{\Delta}$, possibly unbounded.

Assumption 1
$A(\Delta)$, $C(\Delta)$ belong to compact sets when Δ is an admissible uncertainty, (i.e. $\Delta \in \mathbf{\Delta}$).

Stability of uncertain model (1) may be assessed through the very different notions of robust stability or quadratic stability.

Definition 1 *Robust Stability*
(1) is robustly stable if and only if there exists a parameter-dependent symmetric matrix $P(\Delta)$ such that for all admissible uncertainty $\Delta \in \mathbf{\Delta}$:

$$P(\Delta) > \mathbf{0} \qquad A'(\Delta)P(\Delta) + P(\Delta)A(\Delta) < \mathbf{0} \tag{2}$$

Definition 2 *Quadratic Stability*
(1) is quadratically stable if and only if there exists a symmetric matrix P such that $P > \mathbf{0}$ and for all admissible uncertainty $\Delta \in \mathbf{\Delta}$:

$$A'(\Delta)P + PA(\Delta) < \mathbf{0} \tag{3}$$

Even if for some special cases robust stability and quadratic stability are equivalent, quadratic stability with respect to structured uncertainty is a sufficient condition for robust stability. The new conditions proposed in this paper are based on robust stability and are compared to those derived in the quadratic stability framework.

Assumption 2 *The nominal system is stable, i.e. $A = A(\mathbf{0})$ is stable.*

It is to be noted that the condition $P(\Delta) > \mathbf{0}$ can be very hard to verify. Under the slight additional assumption 2 and following lemma 1.1 in [?], this constraint can be removed. Let $T_{w/z}(\Delta)$ be the transfer function of the system for a given uncertainty Δ.

Definition 3
If the system is robustly stable, we define $\mathcal{H}_{w.c.}$, the worst case $\mathcal{H}_2$ cost of the system:

$$\mathcal{H}_{w.c.} = \max_{\Delta \in \mathbf{\Delta}} \|T_{w/z}(\Delta)\|_2 \tag{4}$$

Any upper bound on the worst case $\mathcal{H}_2$ cost of the system, $\mathcal{H} \geq \mathcal{H}_{w.c.}$, is called a robust guaranteed $\mathcal{H}_2$ cost

Here, we focus on state-space methods for deriving tight upper-bounds on $\mathcal{H}_{w.c.}$. A classical way to do so is to use bounds on observability grammian, (respectively controllability grammian). Since

$$\|T_{w/z}(\Delta)\|_2^2 = trace(B'\mathcal{L}_o(\Delta)B)$$

where $\mathcal{L}_o(\Delta)$ is the observability grammian, solution of the Lyapunov equation:

$$A'(\Delta)\mathcal{L}_o(\Delta) + \mathcal{L}_o(\Delta)A(\Delta) + C'(\Delta)C(\Delta) = \mathbf{0}$$

An optimisation problem for computing $\mathcal{H}_{w.c.}$ may be defined as follows:

Problem 1

$$\begin{array}{l} \mathcal{H}_{p_1}^2 = \min \Gamma \\ s.t. \quad \forall\ \Delta \in \mathbf{\Delta} \\ \left\{ \begin{array}{l} A'(\Delta)P(\Delta) + P(\Delta)A(\Delta) + C'(\Delta)C(\Delta) < \mathbf{0} \\ trace(B'P(\Delta)B) < \Gamma \end{array} \right. \end{array} \tag{5}$$

Lemma 1
If the system (1) is robustly stable then $\mathcal{H}_{p_1}$ is the worst case $\mathcal{H}_2$ cost of (1): $\boxed{\mathcal{H}_{p_1} = \mathcal{H}_{w.c.}}$

The optimisation problem 1 is not easily tractable in this form, even though the constraints and the criteria are linear in the decision variables $P(\Delta)$ and Γ.
Another way to tackle the same problem is to introduce new variables that will be usefull in the sequel.

Problem 2

$$\begin{array}{l} \mathcal{H}_{p_2}^2 = \min\ trace(T) \\ s.t. \quad \forall\ \Delta \in \mathbf{\Delta} \\ \left\{ \begin{array}{l} \begin{bmatrix} \mathbf{0} & C'(\Delta) & P(\Delta) \\ C(\Delta) & -\mathbf{1} & \mathbf{0} \\ P(\Delta) & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} A'(\Delta) \\ \mathbf{0} \\ -\mathbf{1} \end{bmatrix} G_1'(\Delta) \\ \quad + G_1(\Delta) \begin{bmatrix} A'(\Delta) \\ \mathbf{0} \\ -\mathbf{1} \end{bmatrix}' < \mathbf{0} \\ \begin{bmatrix} -T & \mathbf{0} \\ \mathbf{0} & P(\Delta) \end{bmatrix} + \begin{bmatrix} B' \\ -\mathbf{1} \end{bmatrix} G_2'(\Delta) + G_2(\Delta) \begin{bmatrix} B' \\ -\mathbf{1} \end{bmatrix}' < \mathbf{0} \end{array} \right. \end{array} \tag{6}$$

Lemma 2
If, the system is robustly stable then $\mathcal{H}_{p_2}$ is the worst-case $\mathcal{H}_2$ cost of (1): $\boxed{\mathcal{H}_{p_2} = \mathcal{H}_{p_1} = \mathcal{H}_{w.c.}}$

The last result states that the problems 1 and 2 are equivalent and lead to the same costs. At this level of generality, these two equivalent problems cannot be solved efficiently. First, more information on the modelling of the uncertainty is needed. Secondly, for a given model of uncertainty, they reduce to infinite-dimensional, sometimes convex, optimisation problems. Finally, a parametrization of the parameter-dependent Lyapunov functions has to be assumed in order to derive tractable conditions on matrix unknowns.

3 Quadratic stability for robust $\mathcal{H}_2$ performance

In this section, the well-known quadratic stability framework is reminded. The optimisation problem 1 becomes:

Problem 3

$$\begin{array}{l} \mathcal{H}_{p_3}^2 = \min \Gamma \\ s.t. \quad \forall\ \Delta \in \mathbf{\Delta} \quad \left\{ \begin{array}{l} trace(B'PB) < \Gamma \\ A'(\Delta)P + PA(\Delta) + C'(\Delta)C(\Delta) < \mathbf{0} \end{array} \right. \end{array} \tag{7}$$

Theorem 1
If, the system is quadratically stable then $\mathcal{H}_{p_3}$ is a robust $\mathcal{H}_2$ guaranteed cost for (1): $\boxed{\mathcal{H}_{w.c.} \leq \mathcal{H}_{p_3}}$

Following the quadratic stability framework we propose the next optimisation problem derived from problem 2:

Problem 4

$$
\mathcal{H}_{p_4}^2 = \min \ trace(T) \\
s.t. \ \ \forall\ \Delta \in \mathbf{\Delta} \\
\left\{\begin{array}{l}
\begin{bmatrix} \mathbf{0} & C'(\Delta) & P \\ C(\Delta) & -\mathbf{1} & \mathbf{0} \\ P & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} A'(\Delta) \\ \mathbf{0} \\ -\mathbf{1} \end{bmatrix} G_1' \\
+G_1 \begin{bmatrix} A'(\Delta) \\ \mathbf{0} \\ -\mathbf{1} \end{bmatrix}' < \mathbf{0} \\
\\
\begin{bmatrix} -T & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} + \begin{bmatrix} B' \\ -\mathbf{1} \end{bmatrix} G_2' + G_2 \begin{bmatrix} B' \\ -\mathbf{1} \end{bmatrix}' < \mathbf{0}
\end{array}\right. \tag{8}
$$

Lemma 3

If, the system is quadratically stable then $\mathcal{H}_{p_4}$ is a robust $\mathcal{H}_2$ guaranteed cost for (1) and moreover it is equal to the solution of problem 3: $\boxed{\mathcal{H}_{w.c.} \leq \mathcal{H}_{p_4} = \mathcal{H}_{p_3}}$

The important point is that problem 2 alongside the previous problem, is particularly appropriate for robust performance analysis based on PDLF and even with some restrictions on the parameter-dependent matrices the optimisation problems allways give better solutions than the guaranteed cost of the quadratic framework. The parametrization of the parameter-dependent matrices has now to be clarified for different modellings of the uncertainty.

4 $\mathcal{LMI}$ conditions for robust $\mathcal{H}_2$ performance analysis

4.1 L.F.T. uncertain models

We assume here that the uncertainty $\Delta \in \mathbf{\Delta}$ enters the model in a linear fractional form:

$$
\begin{array}{l}
A(\Delta) = A - B_\Delta \Delta(\mathbf{1} + D_\Delta \Delta)^{-1} C_\Delta \\
C(\Delta) = C - D_z \Delta(\mathbf{1} + D_\Delta \Delta)^{-1} C_\Delta
\end{array} \tag{9}
$$

This corresponds to a Linear Fractional Transform (L.F.T), and can be seen as a feedback of the uncertainty Δ on the nominal system:

$$
w_\Delta(t) = -\Delta z_\Delta(t) \quad \left\{\begin{array}{l}
\dot{x}(t) = Ax(t) + B_\Delta w_\Delta(t) + Bw(t) \\
z_\Delta(t) = C_\Delta x(t) + D_\Delta w_\Delta(t) \\
z(t) = Cx(t) + D_z w_\Delta(t)
\end{array}\right. \tag{10}
$$

where $\Delta \in \mathbb{R}^{q_\Delta \times p_\Delta}$. The system is supposed to be well-posed [?] and therefore assumption 1 is satisfied. When dealing with L.F.T. uncertain models, a useful tool for robustness analysis is the quadratic separation, [?],[?],[?]. The notion of candidate for quadratic separation is now defined.

Definition 4

For $\Delta \in \mathbf{\Delta} \subset \mathbb{R}^{q \times p}$, $\mathbf{\Theta_\Delta}$ is the set of all candidates for quadratic separation with respect to Δ.

$$
\mathbf{\Theta_\Delta} = \left\{ \Theta = \Theta' \ : \ \begin{bmatrix} \mathbf{1} \\ \Delta \end{bmatrix}' \Theta \begin{bmatrix} \mathbf{1} \\ \Delta \end{bmatrix} \geq \mathbf{0} \ \forall\ \Delta \in \mathbf{\Delta} \right\} \tag{11}
$$

We aim at using the information about the structure of the uncertain model in the parameter-dependent Lyapunov function to reduce the conservatism of the parametrisation proposed in the quadratic stability framework. It is then proposed to look for L.F.T. parameterised Lyapunov functions defined as:

$$
P(\Delta) = \begin{bmatrix} \mathbf{1} \\ -\Delta R(\Delta) \end{bmatrix}' Q \begin{bmatrix} \mathbf{1} \\ -\Delta R(\Delta) \end{bmatrix} \ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} \tag{12}
$$

with $R(\Delta) = (\mathbf{1} + D_\Delta \Delta)^{-1} C_\Delta$. Our objective is to derive tractable conditions for robust $\mathcal{H}_2$ performance analysis using the new general conditions given in problem 2. We have therefore a choice for the structure of $G_1(\Delta)$, $G_2(\Delta)$ to assume. Related to the structure of $P(\Delta)$, the chosen structure is defined as:

$$
\begin{array}{l}
G_2(\Delta) = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & -R'(\Delta)\Delta' \end{bmatrix} F_2 \\
G_1(\Delta) = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & -R'(\Delta)\Delta' & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & -R'(\Delta)\Delta' \end{bmatrix} F_1
\end{array} \tag{13}
$$

where F_1, F_2 are unknown fixed matrices to be determined. These particular choices allow to define a problem which is a particular instance of problem 2. First, consider the following notations:

$$
\overline{\mathbf{\Delta}} = \left\{ \begin{bmatrix} \Delta & \mathbf{0} \\ \mathbf{0} & \Delta \end{bmatrix} \ : \ \Delta \in \mathbf{\Delta} \right\}
$$

$$
\begin{array}{l}
\mathcal{A}_1' = \begin{bmatrix} A' \\ \mathbf{0} \\ -\mathbf{1} \\ B_\Delta' \\ \mathbf{0} \end{bmatrix} \quad \mathcal{C}_1 = \begin{bmatrix} C_\Delta & \mathbf{0} & \mathbf{0} & D_\Delta & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_\Delta & \mathbf{0} & D_\Delta \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{bmatrix} \\
\mathcal{P}_1(Q) = \begin{bmatrix} \mathbf{0} & C' & Q_{11} & \mathbf{0} & Q_{12} \\ C & -\mathbf{1} & \mathbf{0} & D_z & \mathbf{0} \\ Q_{11} & \mathbf{0} & \mathbf{0} & Q_{12} & \mathbf{0} \\ \mathbf{0} & D_z' & Q_{12}' & \mathbf{0} & Q_{22} \\ Q_{12}' & \mathbf{0} & \mathbf{0} & Q_{22} & \mathbf{0} \end{bmatrix} \\
\mathcal{A}_2' = \begin{bmatrix} B' \\ -\mathbf{1} \\ \mathbf{0} \end{bmatrix} \quad \mathcal{C}_2 = \begin{bmatrix} \mathbf{0} & C_\Delta & D_\Delta \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{bmatrix} \\
\mathcal{P}_2(Q,T) = \begin{bmatrix} -T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_{11} & Q_{12} \\ \mathbf{0} & Q_{12}' & Q_{22} \end{bmatrix}
\end{array} \tag{14}
$$

Using these notations and the above parametrisation, we are able to define the following problem:

Problem 5

$$
\mathcal{H}_{p_5}^2 = \min \ trace(T) \\
s.t. \left\{\begin{array}{l}
\mathcal{P}_1(Q) + \mathcal{A}_1' F_1' + F_1 \mathcal{A}_1 + \mathcal{C}_1' \Theta_1 \mathcal{C}_1 < \mathbf{0} \\
\mathcal{P}_2(Q,T) + \mathcal{A}_2' F_2' + F_2 \mathcal{A}_2 + \mathcal{C}_2' \Theta_2 \mathcal{C}_2 < \mathbf{0} \\
\Theta_1 \in \mathbf{\Theta}_{\overline{\mathbf{\Delta}}} \\
\Theta_2 \in \mathbf{\Theta_\Delta}
\end{array}\right. \tag{15}
$$

The constraints $\Theta \in \mathbf{\Theta}_{\overline{\mathbf{\Delta}}}$ and $\Theta \in \mathbf{\Theta_\Delta}$ are not easily handled. Moreover, without any assumption on the nature nor the structure of the set $\mathbf{\Delta}$, it is not surprising to meet with difficulty when dealing with this constraint. For particular classes of uncertainty set $\mathbf{\Delta}$, some numerical tractable relaxations of these conditions are proposed in the literature. The method consists in constructing a relaxed set $\mathbf{\Theta}_{rel.}$ such that $\mathbf{\Theta}_{rel.} \subset \mathbf{\Theta_\Delta}$ and such that the condition $\Theta \in \mathbf{\Theta}_{rel.}$ is easily numerically tractable. For the case of real structured uncertainty, some relaxations are given in [?] for polytopic and dissipative uncertainties. Problem 1 in the quadratic framework writes as problem 3. By considering the L.F.T. parametrisation of the uncertainty model and using the quadratic separation results, it is equivalent to:

Problem 6 *[?]*

$$\mathcal{H}^2_{p_6} = \min \Gamma$$
$$s.t. \begin{cases} trace(B'PB) < \Gamma \\ \begin{bmatrix} A'P+PA & PB_\Delta \\ B'_\Delta P & \mathbf{0} \end{bmatrix} + \begin{bmatrix} C \\ D_z \end{bmatrix}' \begin{bmatrix} C \\ D_z \end{bmatrix} \\ + \begin{bmatrix} C_\Delta & D_\Delta \\ \mathbf{0} & -\mathbf{1} \end{bmatrix}' \Theta \begin{bmatrix} C_\Delta & D_\Delta \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} < \mathbf{0} \\ \Theta \in \mathbf{\Theta_\Delta} \end{cases} \tag{16}$$

Theorem 2
If the system (1) is robustly stable then $\mathcal{H}_{p_5}$ is a robust $\mathcal{H}_2$ guaranteed cost for (1).
If the system (1) is quadratically stable then $\mathcal{H}_{p_6}$ is a robust $\mathcal{H}_2$ guaranteed cost for (1).
Moreover: $\boxed{\mathcal{H}_{w.c.} \le \mathcal{H}_{p_5} \le \mathcal{H}_{p_6}}$

4.2 An interesting particular case

We assume here that the uncertainty $\Delta \in \mathbf{\Delta}$ acts affinely on the model, i.e. $D_\Delta = \mathbf{0}$:

$$A(\Delta) = A - B_\Delta \Delta C_\Delta \qquad C(\Delta) = C - D_z \Delta C_\Delta \tag{17}$$

The parametrisation of the parameter-dependent Lyapunov function becomes a quadratic function of the uncertainty:

$$P(\Delta) = \begin{bmatrix} \mathbf{1} \\ -\Delta C_\Delta \end{bmatrix}' Q \begin{bmatrix} \mathbf{1} \\ -\Delta C_\Delta \end{bmatrix} \tag{18}$$

An important question is to know if it is interesting to consider complicated parameter-dependent Lyapunov function or if the structure of the parameter-dependent Lyapunov function has only to reflect the uncertainty structure. To get more insight on this point, we assume that $\mathbf{\Delta}$ is the convex hull of a finite set of matrices $\{\Delta_1, \cdots, \Delta_N\}$: $\mathbf{\Delta} = co\{\Delta_1, \cdots, \Delta_N\}$
$\mathbf{\Delta}$ is said to be the polytope with N vertices $\Delta_1, \cdots, \Delta_N$. The uncertainty set is compact. The model being affine in Δ, assumption 1 holds. Keeping in mind the affine structure of the uncertainty model, specific optimisation problems avoiding the quadratic separation step and using different parametrisations of the parameter-dependent Lyapunov matrix $P(\Delta)$ may be defined.
Given the polytopic structure of the uncertainty, we first seek for polytopic-type Lyapunov matrices:

$$P_{p_7}(\Delta) = \sum_{i=1}^{N_l} \zeta_i \ P_i \quad \text{when} \quad \Delta = \sum_{i=1}^{N_l} \zeta_i \ \Delta_i \tag{19}$$

This leads to problem 7 defined as:

Problem 7

$$\mathcal{H}_{p_7} = \min \Gamma$$
$$s.t. \quad \forall i = 1 \cdots N$$
$$\begin{cases} \begin{bmatrix} \mathbf{0} & (C - D_z\Delta_i C_\Delta)' & P_i \\ C - D_z \Delta_i C_\Delta & -\mathbf{1} & \mathbf{0} \\ P_i & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ + \begin{bmatrix} (A - B_\Delta \Delta_i C_\Delta)' \\ \mathbf{0} \\ -\mathbf{1} \end{bmatrix} G_1' + [*]' < \mathbf{0} \\ trace(B'P_iB) < \Gamma \end{cases} \tag{20}$$

Another way to reflect the uncertainty structure in the parametrisation of the parameter-dependent Lyapunov matrix $P(\Delta)$ is to look after affine-dependent matrices defined as:

$$P_{p_8}(\Delta) = Q_{11} - Q_{12}\Delta C_\Delta - C'_\Delta \Delta' Q'_{12} \tag{21}$$

We get then:

Problem 8

$$\mathcal{H}_{p_8} = \min \Gamma$$
$$s.t. \quad \forall i = 1 \cdots N$$
$$\begin{cases} \begin{bmatrix} 0 & * & * \\ C - D_z\Delta_i C_\Delta & -1 & * \\ Q_{11} - Q_{12}\Delta_i C_\Delta - C'_\Delta \Delta'_i Q'_{12} & \mathbf{0} & 0 \end{bmatrix} \\ + \begin{bmatrix} (A - B_\Delta \Delta_i C_\Delta)' \\ 0 \\ -1 \end{bmatrix} G_1' + G_1 \begin{bmatrix} (A - B_\Delta \Delta_i C_\Delta)' \\ 0 \\ -1 \end{bmatrix}' < 0 \\ trace(B'(Q_{11} - Q_{12}\Delta_i C_\Delta - C'_\Delta \Delta'_i Q'_{12})B) < \Gamma \end{cases} \tag{22}$$

The problems 7 and 8 are derived from problem 2. Equivalently from problem 1 with the quadratic stability framework, it is possible to derive a problem of the same type:

Problem 9

$$\mathcal{H}_{p_9} = \min \Gamma$$
$$s.t. \quad \forall i = 1 \cdots N$$
$$\begin{cases} \begin{bmatrix} (A - B_\Delta \Delta_i C_\Delta)'P + P(A - B_\Delta \Delta_i C_\Delta) & (C - D_z\Delta_i C_\Delta)' \\ C - D_z \Delta_i C_\Delta & -1 \end{bmatrix} < 0 \\ trace(B'PB) < \Gamma \end{cases} \tag{23}$$

Method	$\mathcal{H}_{w.c.}$	$\mathcal{H}_{p_7}$	$\mathcal{H}_{p_5}$	$\mathcal{H}_{p_8}$	$\mathcal{H}_{p_9}$	$\mathcal{H}_{p_6}$
$\mathcal{H}_2$ cost	0.83	0.836	1.688	1.689	1.759	1.759
computation time		2692 sec	218 sec	23 sec	≤ 2 sec	≤ 2 sec
number of variables		117	167	59	11	39

5 Numerical example

This example is inspired from the system proposed in [?]. In our case we suppose that two parameters are uncertain: the viscous friction coefficient, c, and the mass m_2. Among the numerically tractable problems of this paper, only problems 5 and 6 can deal with the given LFT structure of the model. The related results are given in table 1. By gridding the uncertainty set, it is possible to evaluate the worst-case norm $\mathcal{H}_{w.c.}$ of the system. A rough evaluation of the worst-case $\mathcal{H}_2$ norm is given in table 1.

Method	$\mathcal{H}_{w.c.}$	$\mathcal{H}_{p_5}$	$\mathcal{H}_{p_6}$
$\mathcal{H}_2$ cost	0.47	0.681	0.815
computation time		191 sec	≤ 2 sec

Table 1: LFT modelling

These results show clearly that the use of a parameter-dependent Lyapunov function improves significantly the guaranteed $\mathcal{H}_2$ cost. This is done at the expense of higher computation time. However, the increased computational complexity mainly comes from the quadratic separation which is more complex in problem 5 than it is in problem 6. It is important to note that the number of decision variables of problem 5 is independent upon the number of uncertain parameters, in contrast with the methods proposed in [?], [?].

The worst-case $\mathcal{H}_2$ norm of table 2 is approximately computed by operating a grid on the polytope with 2^3 vertices of the affine model. It encompasses the worst-case $\mathcal{H}_2$ norm of table 1 since the computation is done on the over-bounding set of uncertainties.

6 Conclusion

A new $\mathcal{LMI}$-based robust $\mathcal{H}_2$ performance condition with respect to real LFT uncertainty has been proposed. This test involves parameter-dependent Lyapunov functions depending in a "quadratic-LFT" manner on the parametric uncertainty. It encompasses tests based on quadratic stability techniques. For linear polytopic real uncertainties, different parametrisations of the Lyapunov function are proposed and investigated with respect to the trade-off between the reduction of the conservatism and the computational complexity. This is illustrated on a numerical example.