COURSE ON LMI OPTIMIZATION
WITH APPLICATIONS IN CONTROL
PART I.3

GEOMETRY OF LMI SETS

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January 2005
Geometry of LMI sets

Given $F_i \in S^m$ we want to characterize the shape in $\mathbb{R}^n$ of the LMI set

\[ S = \{ x \in \mathbb{R}^n : F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0 \} \]

Matrix $F(x)$ is PSD iff its diagonal minors $f_i(x)$ are nonnegative

Diagonal minors are multivariate polynomials of indeterminates $x_i$

So the LMI set can be described as

\[ S = \{ x \in \mathbb{R}^n : f_i(x) \geq 0, \ i = 1, \ldots, n \} \]

which is a semialgebraic set

Moreover, it is a convex set
Example of 2D LMI feasible set

\[ F(x) = \begin{bmatrix} 1 - x_1 & x_1 + x_2 & x_1 \\ x_1 + x_2 & 2 - x_2 & 0 \\ x_1 & 0 & 1 + x_2 \end{bmatrix} \succeq 0 \]

Feasible iff all principal minors nonnegative

System of polynomial inequalities \( f_i(x) \geq 0 \)

1st order minors

\[ f_1(x) = 1 - x_1 \geq 0 \]
\[ f_2(x) = 2 - x_2 \geq 0 \]
\[ f_3(x) = 1 + x_2 \geq 0 \]
2nd order minors

\[ f_4(x) = (1 - x_1)(2 - x_2) - (x_1 + x_2)^2 \geq 0 \]
\[ f_5(x) = (1 - x_1)(1 + x_2) - x_1^2 \geq 0 \]
\[ f_6(x) = (2 - x_2)(1 + x_2) \geq 0 \]
3rd order minor

\[ f_7(x) = (1 + x_2)((1 - x_1)(2 - x_2) - (x_1 + x_2)^2) - x_1^2(2 - x_2) \geq 0 \]
LMI feasible set = intersection of
semialgebraic sets $f_i(x) \geq 0$ for $i = 1, \ldots, 7$
Example of 3D LMI feasible set

LMI set

$$S = \{ x \in \mathbb{R}^3 : \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix} \succeq 0 \}$$

arising in SDP relaxation of MAXCUT

Semialgebraic set

$$S = \{ x \in \mathbb{R}^3 : 1 + 2x_1x_2x_3 - (x_1^2 + x_2^2 + x_3^2) \geq 0, \\
x_1^2 \leq 1, x_2^2 \leq 1, x_3^2 \leq 1 \}$$
Intersection of LMI sets

Intersection of LMI feasible sets

\[ F(x) \succeq 0 \quad x_1 \geq -2 \quad 2x_1 + x_2 \leq 0 \]

is also an LMI

\[
\begin{bmatrix}
F(x) & 0 & 0 \\
0 & x_1 + 2 & 0 \\
0 & 0 & -2x_1 - x_2
\end{bmatrix} \succeq 0
\]
Conic representability

LMI sets are convex semialgebraic sets, but are all convex semialgebraic sets representable by LMIs?

A set $X \subset \mathbb{R}^n$ is conic quadratic representable (CQR) if there exist $N$ affine mappings $F_i(x, u)$ s.t.

$$x \in X \iff \exists u : F_i(x, u) = A_i \begin{bmatrix} x \\ u \end{bmatrix} - b_i \succeq_{LMI} 0$$

$$i = 1, \ldots, N$$

A convex function $f : \mathbb{R}^n \to \mathbb{R}$ is CQR if its epigraph

$$\mathcal{E}pi = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\}$$

is CQR
**SDP/LMI representability (1)**

We say that a convex set $X \subset \mathbb{R}^n$ is **SDP** representable if there exists an affine mapping $F(x,u)$ such that

$$x \in X \iff \exists u : F(x,u) \succeq 0$$

In words, if $X$ is the projection of the solution set of the LMI $F(x,u) \succeq 0$ onto the $x$-space and $u$ are additional, or lifting variables.

We say that a convex set $X \subset \mathbb{R}^n$ is **LMI** representable if there exists an affine mapping $F(x)$ such that

$$x \in X \iff F(x) \succeq 0$$

In other words, additional variables $u$ are not allowed.

Similarly, a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is **SDP** or **LMI** representable if its epigraph

$$\mathcal{E}_{pi} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\}$$

is an **SDP** or **LMI** representable set.
The Lorentz, or ice-cream cone

$\mathbb{L}^{n+1} = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1} : \|x\|_2 \leq t \right\}$

is SDP representable as

$\mathbb{L}^{n+1} = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} : \begin{bmatrix} tI_n & x \\ x' & t \end{bmatrix} \succeq 0 \right\}$

As a result, all (convex quadratic) conic representable sets are also SDP representable

$\mathbb{L}^n \subset \mathbb{S}^n_+$

In the sequel we first give a list of conic representable sets (following Ben-Tal and Nemirovski 2000)
SDP/LMI representability (3)

Quadratic forms

The **Euclidean norm** \( \{ x, t \in \mathbb{R}^n \times \mathbb{R} : \|x\|_2 \leq t \} \)
is **CQR** by definition.

The **squared Euclidean norm**

\[ \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : x'x \leq t \right\} \]
is **CQR** as

\[ \left\| \begin{bmatrix} x \\ t - \frac{1}{2} \end{bmatrix} \right\|_2 \leq \frac{t + \frac{1}{2}}{2} \]
More generally, the **convex quadratic** set

\[
\left\{ x \in \mathbb{R}^n, t \in \mathbb{R} : x'Ax + b'x + c \leq 0 \right\}
\]

with \( A = A' \succeq 0 \) is **CQR** as

\[
\left\| \begin{bmatrix}
    Dx \\
    \frac{t + \frac{b'x + c}{2}}{2}
\end{bmatrix}
\right\|_2 \leq \frac{t - \frac{b'x - c}{2}}{2}
\]

where \( D \) is the **Cholesky factor** of \( A = D'D \)
The branch of hyperbola

\[ \{(x, y) \in \mathbb{R}^2 : xy \geq 1, x > 0\} \]

is CQR as

\[ \left\| \begin{bmatrix} \frac{x-y}{2} \\ 1 \end{bmatrix} \right\|_2 \leq \frac{x+y}{2} \]
SDP/LMI representability (6)
Geometric mean of two variables

The hypograph of the geometric mean of 2 variables

\[ \left\{ (x_1, x_2, t) \in \mathbb{R}^3 : x_1, x_2 \geq 0, \sqrt{x_1 x_2} \geq t \right\} \]

is CQR as

\[ \exists u : u \geq t, \left\| \begin{bmatrix} u \\ \frac{x_1 - x_2}{2} \end{bmatrix} \right\|_2 \leq \frac{x_1 + x_2}{2} \]
SDP/LMI representability (7)
Geometric mean of several variables

The hypograph of the geometric mean of $2^k$ variables

$$\left\{ (x_1, \ldots, x_{2^k}, t) \in \mathbb{R}^{2^k+1} : x_i \geq 0, (x_1 \cdots x_{2^k})^{1/2^k} \geq t \right\}$$

is also CQR

Proof: Iterate the previous construction

Example with $k = 3$:

$$\sqrt{x_{01}x_{02}} \geq x_{11}$$
$$\sqrt{x_{03}x_{04}} \geq x_{12}$$
$$\sqrt{x_{05}x_{06}} \geq x_{13}$$
$$\sqrt{x_{07}x_{08}} \geq x_{14}$$

$$\left\{ \sqrt{x_{11}x_{12}} \geq x_{21} \right\}$$
$$\left\{ \sqrt{x_{13}x_{14}} \geq x_{22} \right\}$$

$$\sqrt{x_{21}x_{22}} \geq x_{31} \geq t$$

Useful idea in other SDP representability problems
Using similar ideas, we can show that the increasing rational power functions

\[ f(x) = x^{p/q}, \quad x \geq 0 \]

with rational \( p/q \geq 1 \), as well as the decreasing

\[ g(x) = x^{-p/q}, \quad x \geq 0 \]

with rational \( p/q \geq 0 \), are both CQR
Example:

\[
\{(x, t) \in \mathbb{R}^2 : x \geq 0, \, x^{7/3} \leq t\}
\]

Start from conic representable

\[
\hat{t} \leq (\hat{x}_1 \cdots \hat{x}_8)^{1/8}
\]

and replace

\[
\begin{align*}
\hat{t} &= \hat{x}_1 = x \geq 0 \\
\hat{x}_2 &= \hat{x}_3 = \hat{x}_4 = t \geq 0 \\
\hat{x}_5 &= \hat{x}_6 = \hat{x}_7 = \hat{x}_8 = 1
\end{align*}
\]

to get

\[
\begin{align*}
x &\leq x^{1/8} t^{3/8} \\
x^{7/8} &\leq t^{3/8} \\
x^{7/3} &\leq t
\end{align*}
\]

Same idea works for any rational \( p/q \geq 1 \)

- lift = use additional variables, and
- project in the space of original variables
SDP/LMI representability (10)
Even power monomial (1)

The epigraph of even power monomial

$$\mathcal{E}_{pi} = \{x, t : x^{2p} \leq t\}$$

where $p$ is a positive integer, is CQR

Note that

$$\{x, t : x^{2p} \leq t\}$$

$$\iff$$

$$\{x, y, t : x^2 \leq y\}$$

$$\{x, y, t : y \geq 0, y^p \leq t\}$$

both CQR

Use lifting $y$ and project back onto $x, t$

Similarly, even power polynomials are CQR (combinations of monomials)
\[ \mathcal{E}_{pi} = \left\{ x, t : x^4 \leq t \right\} \]
SDP/LMI representability (12)
Largest eigenvalue

The epigraph of the function largest eigenvalue of a symmetric matrix

\[
\left\{ X = X' \in \mathbb{R}^{n \times n}, t \in \mathbb{R} : \lambda_{\text{max}}(X) \leq t \right\}
\]

is SDP (LMI) representable as

\[ X \preceq tI_n \]

Eigenvalues of matrix \[
\begin{bmatrix}
1 & x_1 \\
-x_1 & x_2
\end{bmatrix}
\]
SDP/LMI representability (13)
Sums of largest eigenvalues

Let

\[ S_k(X) = \sum_{i=1}^{k} \lambda_i(X), \quad k = 1, \ldots, n \]

denote the sum of the \( k \) largest eigenvalues of \( X \in S^n \).

The epigraph

\[ \left\{ X \in S^{n \times n}, t \in \mathbb{R} : S_k(X) \leq t \right\} \]

is SDP representable as

\[
\begin{align*}
 t - ks - \text{trace } Z &\geq 0 \\
 Z &\succeq 0 \\
 Z - X + sI_n &\succeq 0
\end{align*}
\]

where \( Z \) and \( s \) are additional variables.
Determinant of a PSD matrix

The determinant

\[
\det(X) = \prod_{i=1}^{n} \lambda_i(X)
\]

is not a convex function of \(X\), but the function

\[
f_q(X) = -\det^q(X), \quad X = X' \succeq 0
\]

is convex when \(q \in [0, 1/n]\) is rational

The epigraph

\[
\{f_q(X) \leq t\}
\]

is SDP representable as

\[
\begin{bmatrix}
X & \Delta \\
\Delta' & \text{diag } \Delta
\end{bmatrix} \succeq 0 \\
t \leq (\delta_1 \cdots \delta_n)^q
\]

since we know that the latter constraint (hypograph of a concave monomial) is conic representable

Here \(\Delta\) is a lower triangular matrix of additional variables with diagonal entries \(\delta_i\)
Application: extremal ellipsoids

Various representations of an ellipsoid in $\mathbb{R}^n$

\[
E = \{ x \in \mathbb{R}^n : x'Px + 2x'q + r \leq 0 \} = \{ x \in \mathbb{R}^n : (x - x_c)'P(x - x_c) \leq 1 \} = \{ x = Qy + x_c \in \mathbb{R}^n : y'y \leq 1 \} = \{ x \in \mathbb{R}^n : \|Rx - x_c\| \leq 1 \}
\]

where $Q = R^{-1} = P^{-1/2} \succ 0$

Volume of ellipsoid $E = \{ Qy + x_c : y'y \leq 1 \}$

\[
\text{vol } E = k_n \det Q
\]

where $k_n$ is volume of $n$-dimensional unit ball

\[
k_n = \begin{cases} 
\frac{2^{(n+1)/2} \pi^{(n-1)/2}}{n(n-2)!!} & \text{for } n \text{ odd} \\
\frac{2\pi^{n/2}}{n(n/2-1)!} & \text{for } n \text{ even}
\end{cases}
\]

<table>
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<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>$k_n$</td>
<td>2.00</td>
<td>3.14</td>
<td>4.19</td>
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<td><strong>5.26</strong></td>
<td>5.17</td>
<td>4.72</td>
<td>4.06</td>
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Unit ball has maximum volume for $n = 5$!
Outer and inner ellipsoidal approximations

Let $S \subset \mathbb{R}^n$ be a solid $= a$ closed bounded convex set with nonempty interior

- the largest volume ellipsoid $E_{in}$ contained in $S$ is unique and satisfies

$$E_{in} \subset S \subset nE_{in}$$

- the smallest volume ellipsoid $E_{out}$ containing $S$ is unique and satisfies

$$E_{out}/n \subset S \subset E_{out}$$

These are Löwner-John ellipsoids

Factor $n$ reduces to $\sqrt{n}$ if $S$ is symmetric

How can these ellipsoids be computed?
Ellipsoids and polytopes (1)

Let the intersection of hyperplanes

\[ S = \{ x \in \mathbb{R}^n : a_i'x \leq b_i, \ i = 1, \ldots, m \} \]

describe a polytope

The largest volume ellipsoid contained in \( S \) is

\[ E = \{ Qy + x_c : y'y \leq 1 \} \]

where \( Q, x_c \) are optimal solutions of the LMI

\[
\begin{align*}
\max & \quad \det^{1/n}Q \\
Q & \succeq 0 \\
\|Qa_i\|_2 & \leq b_i - a_i'x_c, \quad i = 1, \ldots, m
\end{align*}
\]
Ellipsoids and polytopes (2)

Let the convex hull of vertices
\[ S = \operatorname{co} \{x_1, \ldots, x_m\} \]
describe a polytope

The smallest volume ellipsoid containing \( S \) is
\[ E = \{ x : (x - x_c)'P(x - x_c) \leq 1 \} \]
where \( P, x_c = -P^{-1}q \) are optimal solutions of the LMI

\[
\begin{align*}
\max & \quad t \\
\text{s.t.} & \quad t \leq \det^{1/n} P \\
& \quad \begin{bmatrix} P & q \\ q' & r \end{bmatrix} \succeq 0 \\
& \quad x_i'Px_i + 2x_i'q + r \leq 1, \quad i = 1, \ldots, m
\end{align*}
\]
SDP representability and singular values

Let

\[ \Sigma_k(X) = \sum_{i=1}^{k} \sigma_i(X), \quad k = 1, \ldots, n \]

denote the sum of the \( k \) largest singular values of \( X \in \mathbb{R}^{n \times n} \).

Then the epigraph

\[ \{ X \in S^n, t \in \mathbb{R} : \Sigma_k(X) \leq t \} \]

is \textit{SDP representable} since

\[ \sigma_i(X) = \lambda_i \left( \begin{bmatrix} 0 & X' \\ X & 0 \end{bmatrix} \right) \]

and the sum of largest eigenvalues of a symmetric matrix is \textit{SDP representable}. 
Nonlinear matrix inequalities (1)
Schur complement

We can use the **Schur complement** to convert a non-linear matrix inequality into an LMI

\[
\begin{align*}
A(x) - B(x)C^{-1}(x)B'(x) &\succeq 0 \\
C(x) &\succ 0
\end{align*}
\]

\[\iff\]

\[
\begin{bmatrix}
A(x) & B(x) \\
B(x) & C(x)
\end{bmatrix} \succeq 0
\]

\[
C(x) \succ 0
\]

Issai Schur
(1875 Mogilyov - 1941 Tel Aviv)
Nonlinear matrix inequalities (2)

Elimination lemma

To remove decision variables we can use the elimination lemma

\[ A(x) + B(x)XC(x) + C'(x)X'B'(x) > 0 \]

\[ \iff \]

\[ B^\perp(x)A(x)B'^\perp(x) > 0 \quad C'^\perp(x)A(x)C'^\perp'(x) > 0 \]

where \( B^\perp \) and \( C'^\perp \) are orthogonal complements of \( B \) and \( C' \) respectively, and \( x \) is a decision variable independent of matrix \( X \)

Can be shown with SDP duality and theorem of alternatives
The set of univariate polynomials that are positive on the real axis is a convex set that is LMI representable.

Can be proved with cone duality (Nesterov) or with theory of moments (Lasserre).

The even polynomial

\[ p(s) = p_0 + p_1 s + \cdots + p_{2n} s^{2n} \]

satisfies \( p(s) \geq 0 \) for all \( s \in \mathbb{R} \) if and only if

\[
\begin{align*}
p_k &= \sum_{i+j=k} X_{ij}, & k = 0, 1, \ldots, 2n \\
&= \text{trace } H_k X
\end{align*}
\]

for some matrix \( X = X' \succeq 0 \).
The expression of $p_k$ with Hankel matrices $H_k$ comes from

$$p(s) = [1 \ s \ \cdots \ s^n]X[1 \ s \ \cdots \ s^n]^*$$

hence $X \succeq 0$ naturally implies $p(s) \geq 0$

Conversely, existence of $X$ for any polynomial $p(s) \geq 0$ follows from the existence of a sum-of-squares decomposition (with at most two elements) of

$$p(s) = \sum_k q_k^2(s) \geq 0$$

Matrix $X$ has entries $X_{ij} = \sum_k q_k i q_k j$
Optimizing over polynomials (1)
Primal and dual formulations

Global minimization of polynomial

\[ p(s) = \sum_{k=0}^{n} p_k s^k \]

Global optimum \( p^* \): maximum value of \( \hat{p} \) such that \( p(s) - \hat{p} \geq 0 \) for all \( s \in \mathbb{R} \)

Primal LMI

\[
\begin{align*}
\text{max} \quad & \hat{p} = p_0 - \text{trace } H_0 X \\
\text{s.t.} \quad & \text{trace } H_k X = p_k, \quad k = 1, \ldots, n \\
& X \succeq 0
\end{align*}
\]

Dual LMI

\[
\begin{align*}
\text{min} \quad & p_0 + \sum_{k=1}^{n} p_k y_k \\
\text{s.t.} \quad & H_0 + \sum_{k=1}^{n} H_k y_k \succeq 0
\end{align*}
\]

with Hankel structure (moment matrix)
Optimizing over polynomials (2)

Example

Global minimization of the polynomial

\[ p(s) = 48 - 92s + 56s^2 - 13s^3 + s^4 \]

We just have to solve the dual LMI

\[
\begin{align*}
\min & \quad 48 - 92y_1 + 56y_2 - 13y_3 + y_4 \\
\text{s.t.} & \quad \begin{bmatrix} 1 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix} \succeq 0
\end{align*}
\]

to obtain \( p^* = p(5.25) = -12.89 \)
Complex LMIs

The complex valued LMI

\[ F(x) = A(x) + jB(x) \succeq 0 \]

is equivalent to the real valued LMI

\[
\begin{bmatrix}
A(x) & B(x) \\
-B(x) & A(x)
\end{bmatrix} \succeq 0
\]

If there is a complex solution to the LMI then there is a real solution to the same LMI.

Note that matrix \( A(x) = A'(x) \) is symmetric whereas \( B(x) = -B'(x) \) is skew-symmetric.
Rigid convexity

Helton & Vinnikov showed that a convex 2D set

\[ \mathcal{F} = \{ x \in \mathbb{R}^2 : p(x) \geq 0 \} \]

defined by a polynomial \( p(x) \) of minimum degree \( d \) is LMI representable without lifting variables iff \( \mathcal{F} \) is rigidly convex, meaning that

for every point \( x \in X \) and almost every line through \( x \)
then the line intersects \( p(x) = 0 \) in exactly \( d \) points

Example: \( \mathcal{F} = \{(x_1, x_2 \in \mathbb{R}^2 : p(x) = x_2 - x_1^4 \geq 0\} \)
with 2 line intersections
is not rigidly convex because \( 2 < d = 4 \)

.. but it is LMI representable with lifting variables
see the previous construction for even power monomials