Understanding, Exploiting and Extending SAT solvers

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With the help of

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for the practical Session

Today's Itinerary

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Today's Roadmap: learns & practice

Learn the algorithms behind modern SAT solvers

Practice with the 100% pure python SAT Solver pysat bitbucket

Play with it in python (your favorite editor) or with the Jupyter Notebooks

You have a few minutes to check your Python3 / Jupyter installation

Go to
http://www.labri.fr/perso/lsimon/toulouse2018
to get ready

Why studying SAT?

SAT, the canonical NP-Complete problem.

Related to a one-million dollar question (is NP=P?)

- The **main open problem** of Theoretical Computer Science
- **Must be faced** in most real-world problems
- But, the *easiest of the hard* problems ⇐ **Hope here!**

S. Aaronson, MIT : « **If P = NP**, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in 'creative leaps,' no fundamental gap between solving a problem and recognizing the solution once it's found. **Everyone who could appreciate a symphony would be Mozart; everyone who could follow a step-by-step argument would be Gauss.** »[1]

[1] citation taken from [Vardi, 2015]

Why studying SAT? – 2

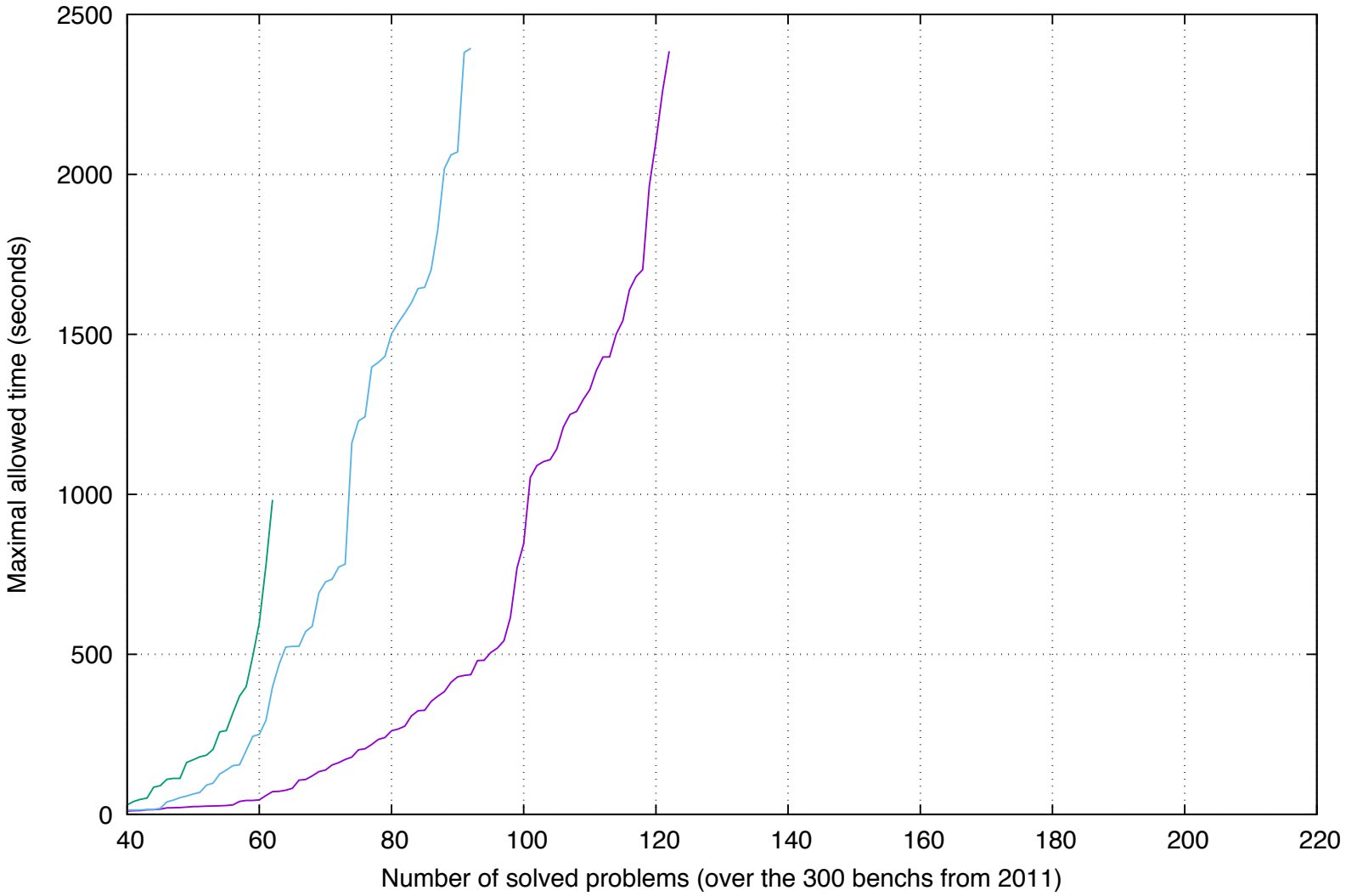
Reduction to SAT: **prove the hardness of new problems**

- **many reductions exist from many problems to SAT**
- **SAT "captures" the difficulty** of many other problems

SAT Solvers are (incredibly) efficients and "user-friendly"

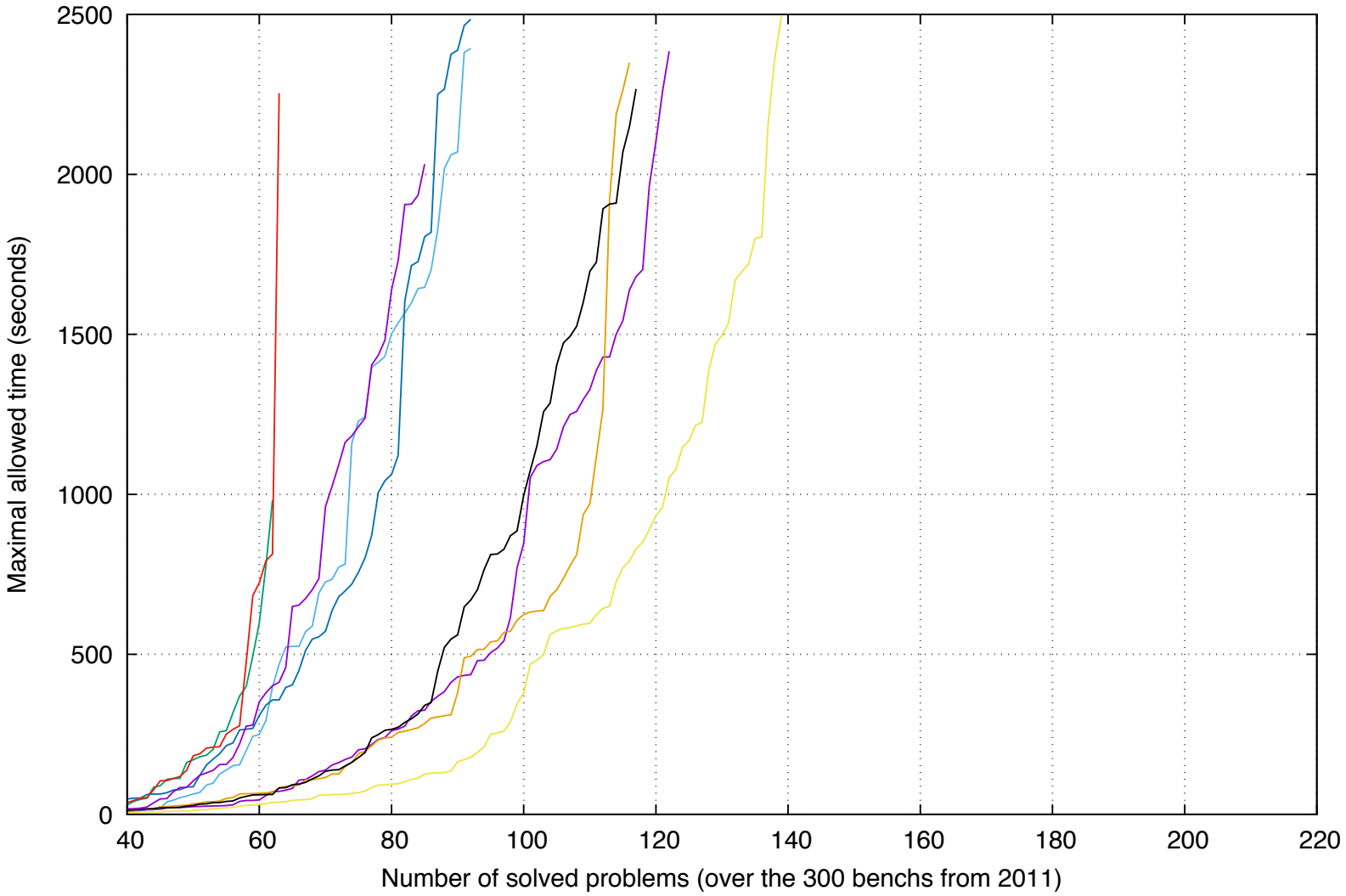
- Can be used as a **black box**
- Can be used as an **open box** (special heuristics, ...)
- Can be **extended** in many ways
  - SAT solvers as Oracles on many close formulas
  - SAT solvers working at an abstract level
  - SAT solvers working on "on the fly" generated constraints
  - ...

More and more efficient SAT solving



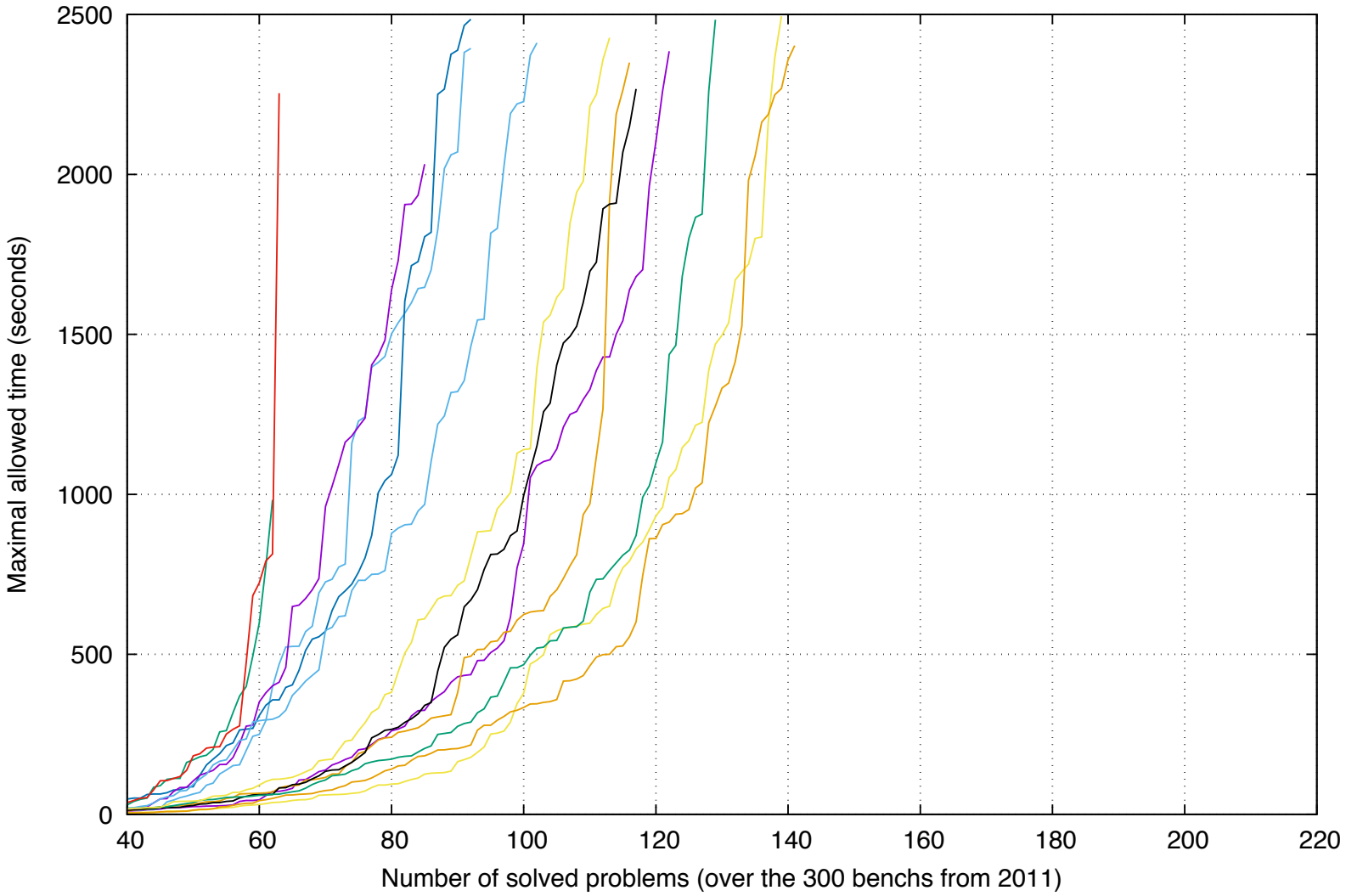
2002

More and more efficient SAT solving



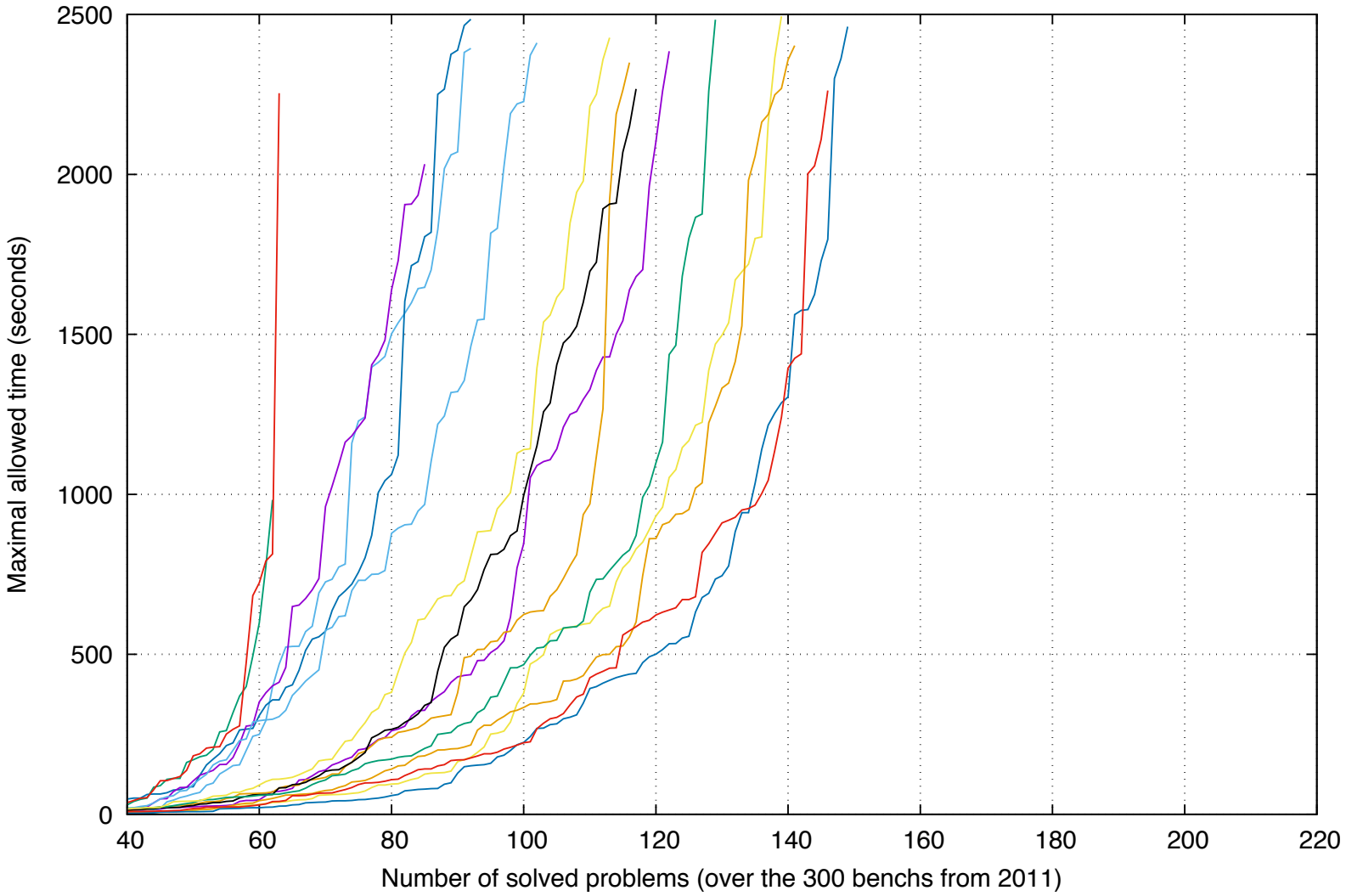
2003

More and more efficient SAT solving



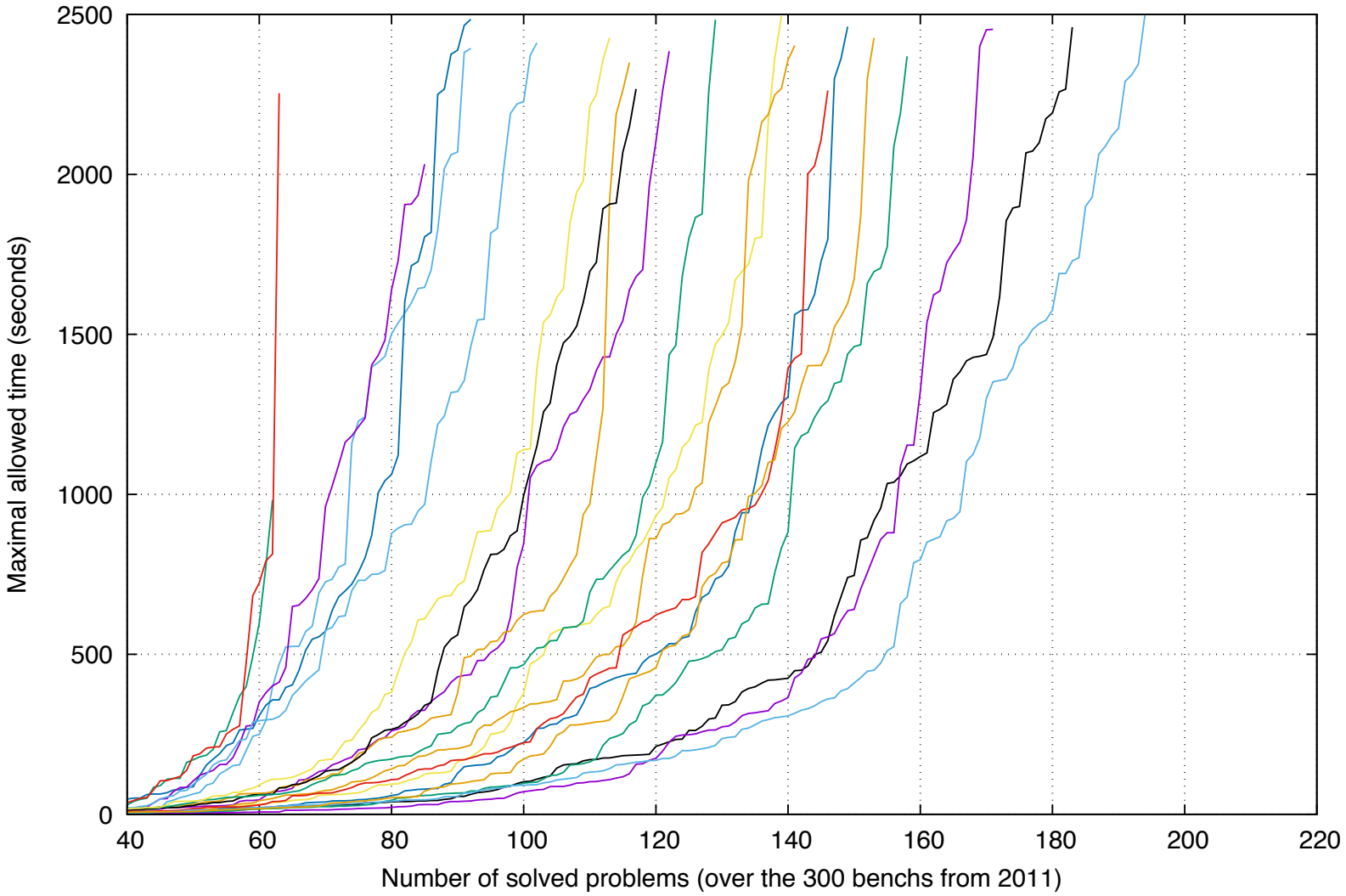
2005

More and more efficient SAT solving



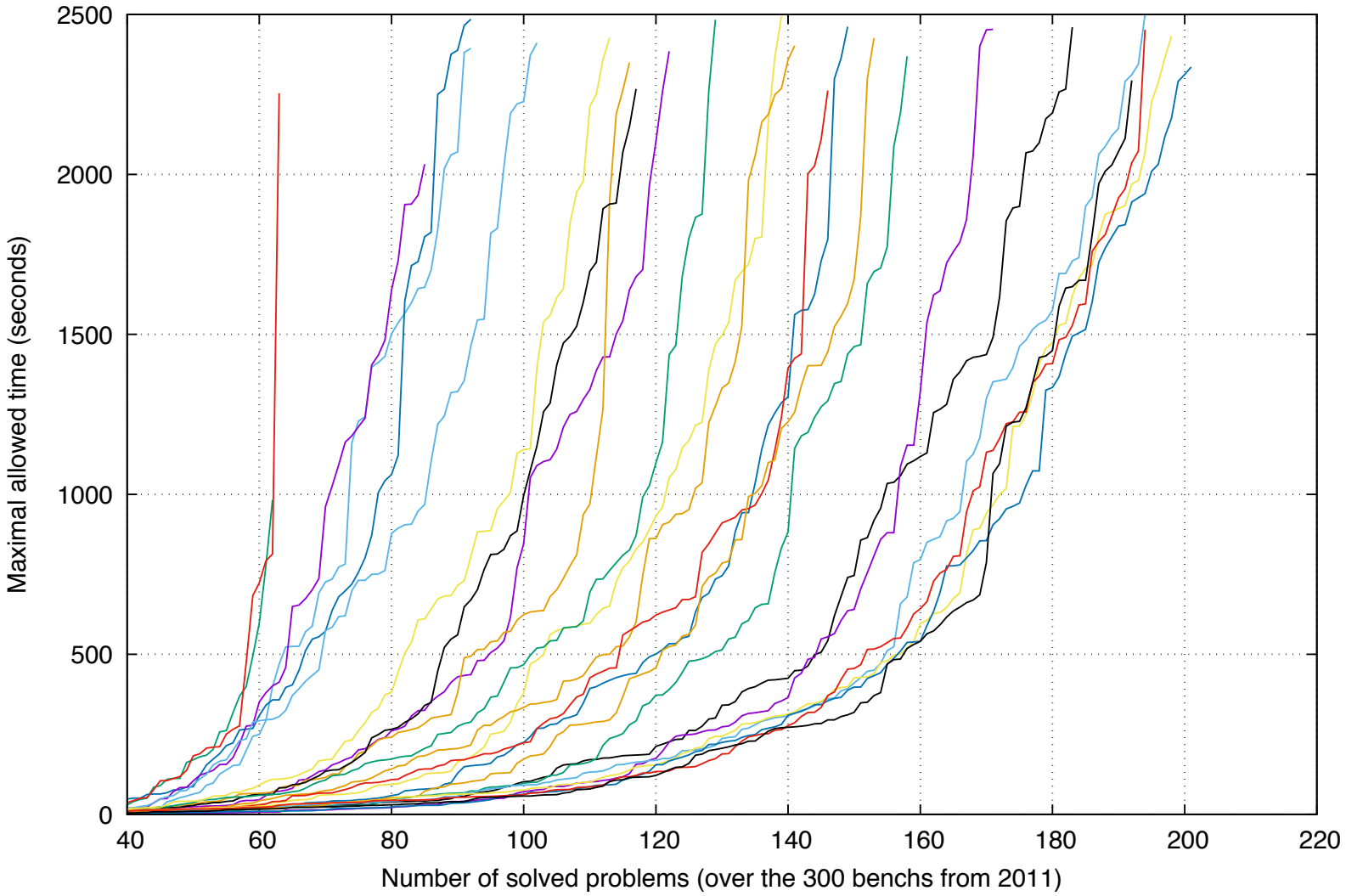
2007

More and more efficient SAT solving



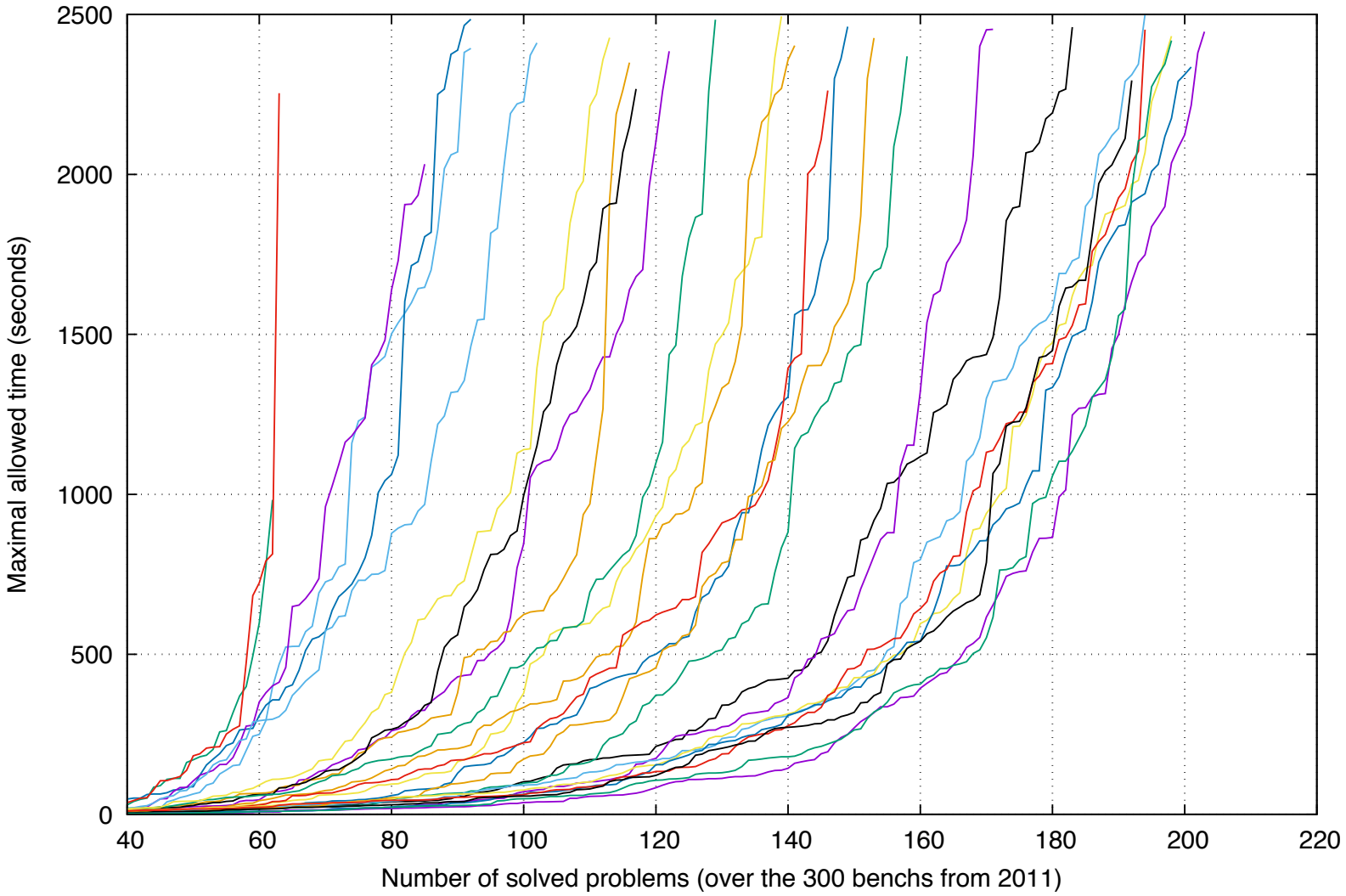
2009

More and more efficient SAT solving



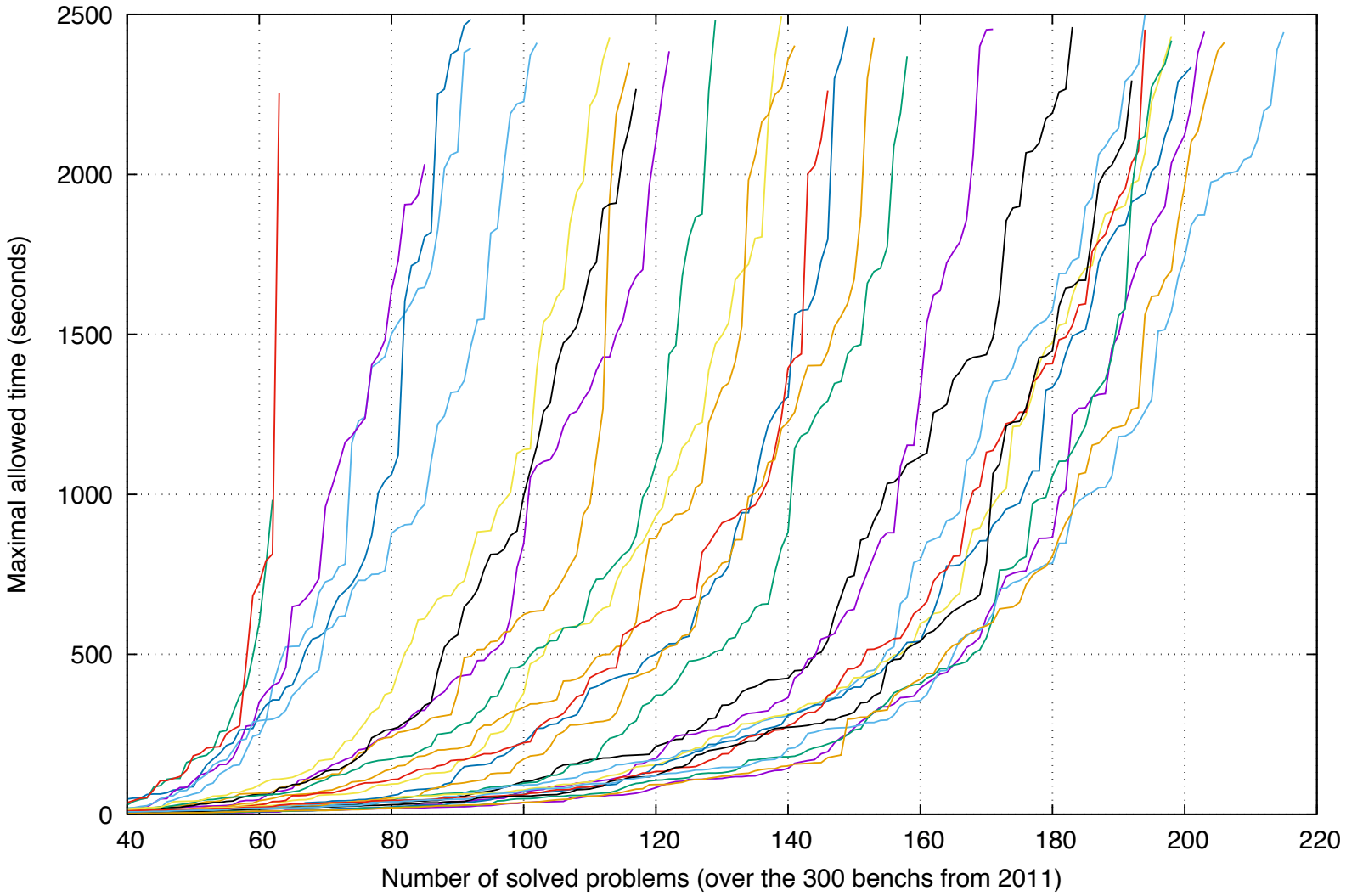
2011

More and more efficient SAT solving



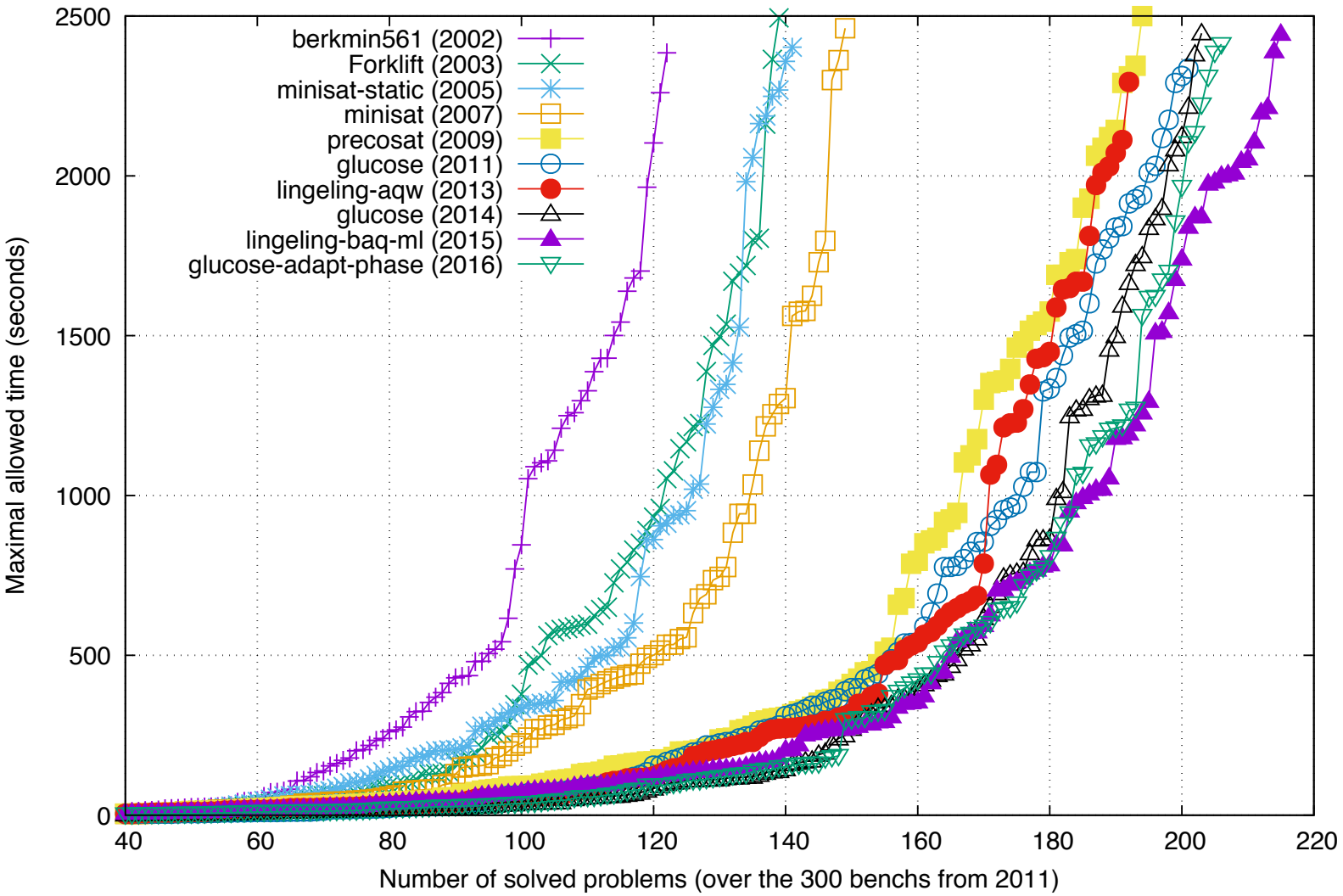
2014

More and more efficient SAT solving



2016

More and more efficient SAT solving



the winners

Today's Itinerary

Propositional Logic: « Simple is beautiful »

Knowledge Representation

The **facts**:
are true, or false.

The **variables**:
are $\top$, or $\bot$.

Reasoning by calculus

If we know that:

- A implies B
- B implies C

Then, we can **deduce**:

- A implies C

If we have :

- $\neg A \vee B$
- $\neg B \vee C$

Then, by **resolution** :

- $\neg A \vee C$

Logic defined more than 2300 years ago!

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Simple is beautiful... But speed is essential

Is it really interesting to study how to implement a CDCL?

The paradigm shift is essentially due to

- **Tight data structures**
- **Algorithms built upon this data structure**

→ [The way "Modern" SAT solvers are solving problems has nothing common with a human strategy]

Interesting problems are not toy problems

Beeing fast is the way computers are not so dumb

What can be done with a so simple logic?

The facts are propositional variables
The knowledge is a propositional formula

$$
\begin{array}{lrcrcr}
 & \neg x_1 & \vee & \neg x_2 & \vee & x_3 \\
\wedge & & & & & \neg x_3 \\
\wedge & x_1 & \vee & x_2 & & \\
\wedge & & & x_2 & \vee & x_3
\end{array}
$$

- Variables : $x_1 \ldots x_3$;
- Literals : x_1, $\neg x_1$;
- Clauses : $\neg x_1 \vee \neg x_2 \vee x_3$;
- Formula Σ written in CNF (conjonction of clauses);

Big questions

- **SAT** : is there an assignment of variables making the formula true?
- **UNSAT**: is the theory contradictory?
- **PI** : deduce all you can from Σ

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$\bot$	$\bot$	$\bot$

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The SAT problem

Definition

Input: A set of clauses built from a propositional language with n variables.

Output: Is there an assignment of the n variables that satisfies all those clauses?

Example (with different notations)

$$\Sigma_1 = (\neg a \vee b) \wedge (\neg b \vee c) = (a' + b).(b' + c) = \{\neg a \vee b, \neg b \vee c\}$$

$$\Sigma_2 = \Sigma_1 \wedge a \wedge \neg c = \Sigma_1 \cup \{a, \neg c\}$$

For Σ_1, the answer is yes, for Σ_2 the answer is no

$$\Sigma_1 \models \neg a \vee c = \neg(a \wedge \neg c)$$

Why working on CNF? Bec(l)ause!

Why considering SAT only CNF only?

(SAT is trivial on DNF!)

- Human-designed systems are conjunctions of properties (in general)
- We want to do symbolic reasoning (not trying all possible solutions)
- There are (too) many ways of applying rules at each step
- We need to restrict the possibilities at each step

But, what about rewritting any formula into CNF?

- Very hard in general
- But, very easy *if we just want to check SAT*

Working on CNF [Tseitin 1968]

Idea: introduce new variables encoding the satifiability of subformulas

Let's say we need to check $SAT(f)$ with

$$f \equiv g \vee h$$

We introduce x_f, x_g and x_h representing the satifiability of f, g and h, respectively

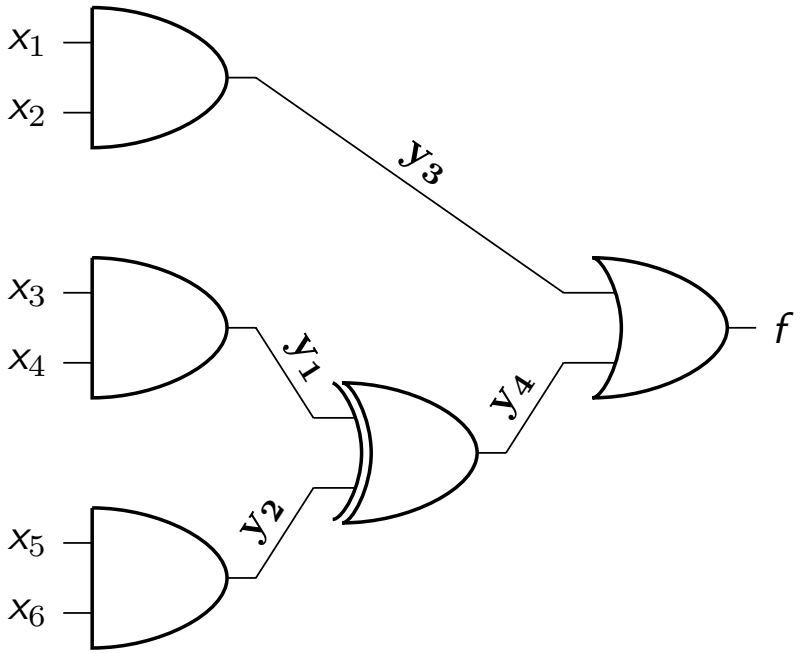
$$(\neg x_f \vee x_g \vee x_h) \wedge (x_f \vee \neg x_g) \wedge (x_f \vee \neg x_h)$$

x_f encodes the satisfiability of f. Easy! (linear, no blow-up!)

(Introducing new variables is so powerful, isn't it?)

It is like naming all the wires in a circuit

$$f = (x_1 \wedge x_2) \vee ((x_3 \wedge x_4) \oplus (x_5 \wedge x_6))$$



Adds y_1, y_2, y_3, y_4, y_f and:

$$\Sigma_f \equiv \begin{pmatrix} (y_f \leftrightarrow y_3 \wedge y_4) \\ \wedge\, (y_4 \leftrightarrow y_1 \oplus y_2) \\ \wedge\, (y_1 \leftrightarrow x_3 \wedge x_4) \\ \wedge\, (y_2 \leftrightarrow x_5 \wedge x_6) \\ \wedge\, (y_f) \end{pmatrix}$$

f is satisfiable iff Σ_f is

How can we solve it?

- Try all possible solutions?
- Try to guess a solution?
- Any other idea?

In this Master Class we will not consider local search (we can prove UNSAT!)

At the heart of most procedures: resolution

The Resolution Rule (Cut) [Gentzen 1934, Robinson 1965]

Let $c_1 = (x \vee a_1 \vee \ldots a_n)$ and $c_2 = (\neg x \vee b_1 \vee \ldots b_m)$

$$c = (a_1 \vee \ldots a_n \vee b_1 \vee \ldots b_m)$$

is obtained by resolution on x between c_1 and c_2

It is a particular case of the following **deduction rule**:

if $a \rightarrow b$ **and** $b \rightarrow c$ **then** $a \rightarrow c$

In general, SAT solvers are only using this rule
(but many, many times per second)

Finding which ones to trigger is the secret of efficient SAT solvers

Today's Itinerary

Davis & Putnam: the firsts SAT steps

1958: Hilary Putnam and Martin Davis look for funding their research around propositional logic

« What we're interested in is good algorithms for propositional calculus » (NSA)

Before that, only inefficient methods (truth tables, ...)

First papers

- *Computational Methods in The Propositional calculus* [Davis Putnam 1958][2]
- *A Computing Procedure for Quantification Theory* [Davis Putnam 1960]

[2] Rapport interne NSA

1960, already a first (kind of) competition!

« *The superiority of the present procedure (i.e. DP) over those previously available is indicated in part by the fact that a formula on which Gilmore's routine for the IBM 704 causes* **the machine to compute for 21 minutes** *without obtaining a result was worked successfully by* **hand computation using the present method in 30 minutes** »

[Davis et Putnam 1960], page 202.

Principles of DP-60

DP-60: forgets variables one after the other

Example : forgets x_1.

$$
\begin{array}{l}
x_1 \vee x_4 \\
\overline{x_1} \vee x_4 \vee x_{14} \\
\overline{x_1} \vee \overline{x_3} \vee \overline{x_8} \\
x_1 \vee x_8 \vee x_{12} \\
x_1 \vee x_5 \vee \overline{x_9} \\
x_2 \vee x_{11} \\
\overline{x_3} \vee \overline{x_7} \vee x_{13} \\
\overline{x_3} \vee \overline{x_7} \vee \overline{x_{13}} \vee x_9 \\
x_8 \vee \overline{x_7} \vee \overline{x_9}
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$\overline{x_1} \vee \overline{x_3} \vee \overline{x_8}$

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$$x_1 \vee \left(\begin{array}{l} x_4 \\ x_8 \vee x_{12} \\ x_5 \vee \overline{x_9} \end{array} \right)$$

$$\overline{x_1} \vee \left(\begin{array}{l} x_4 \vee x_{14} \\ \overline{x_3} \vee \overline{x_8} \end{array} \right)$$

$$\begin{array}{l} x_2 \vee x_{11} \\ \overline{x_3} \vee \overline{x_7} \vee x_{13} \\ \overline{x_3} \vee \overline{x_7} \vee \overline{x_{13}} \vee x_9 \\ x_8 \vee \overline{x_7} \vee \overline{x_9} \end{array}$$

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Variable elimination – on a trivial example

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \models$$

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$$x_3 \quad \models$$

$$\top$$

- Formula is SAT (and x_3 is True in all models)

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$$(\neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \models$$

$$(x_3 \vee \neg x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \models$$

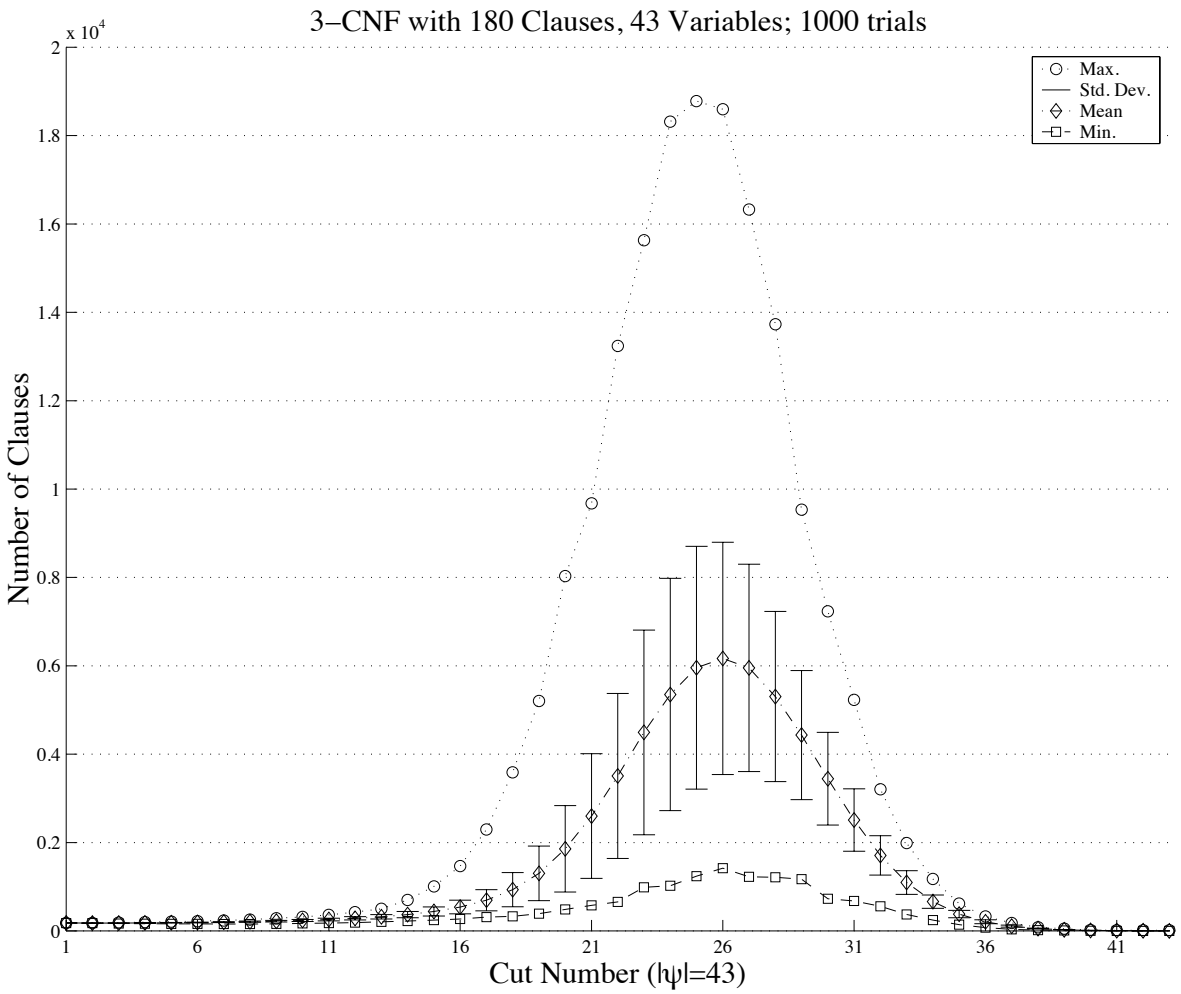
$$(x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \models$$

$$x_3 \quad \models$$

$$\top$$

- Formula is SAT (and x_3 is True in all models)

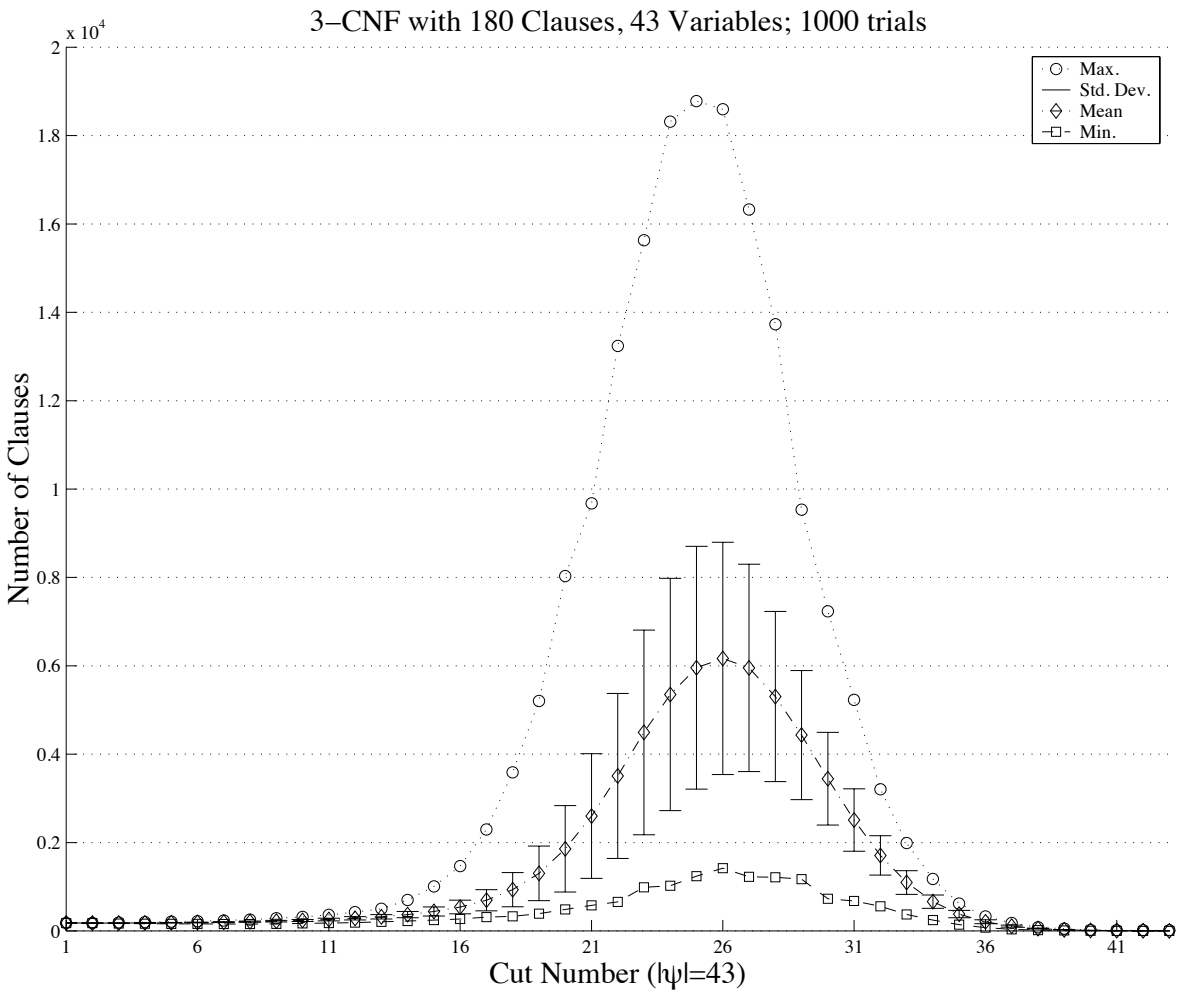
Untractable Space Problems



Combinatorial explosion, even on very small problems!

But possible on some very special cases (SAT pre-processing)

Untractable Space Problems



Combinatorial explosion, even on very small problems!

But possible on some very special cases (SAT pre-processing)

1962-2001 : DPLL rules the world

Systematically explore the space of partial models (backtrack)

- Choose a literal
- Try to find a solution with this literal set to True
- If it is not possible:
  Finds a solution with this literal set to False

Backtrack search on partial models
Systematic (ordered) exploration ensures completeness

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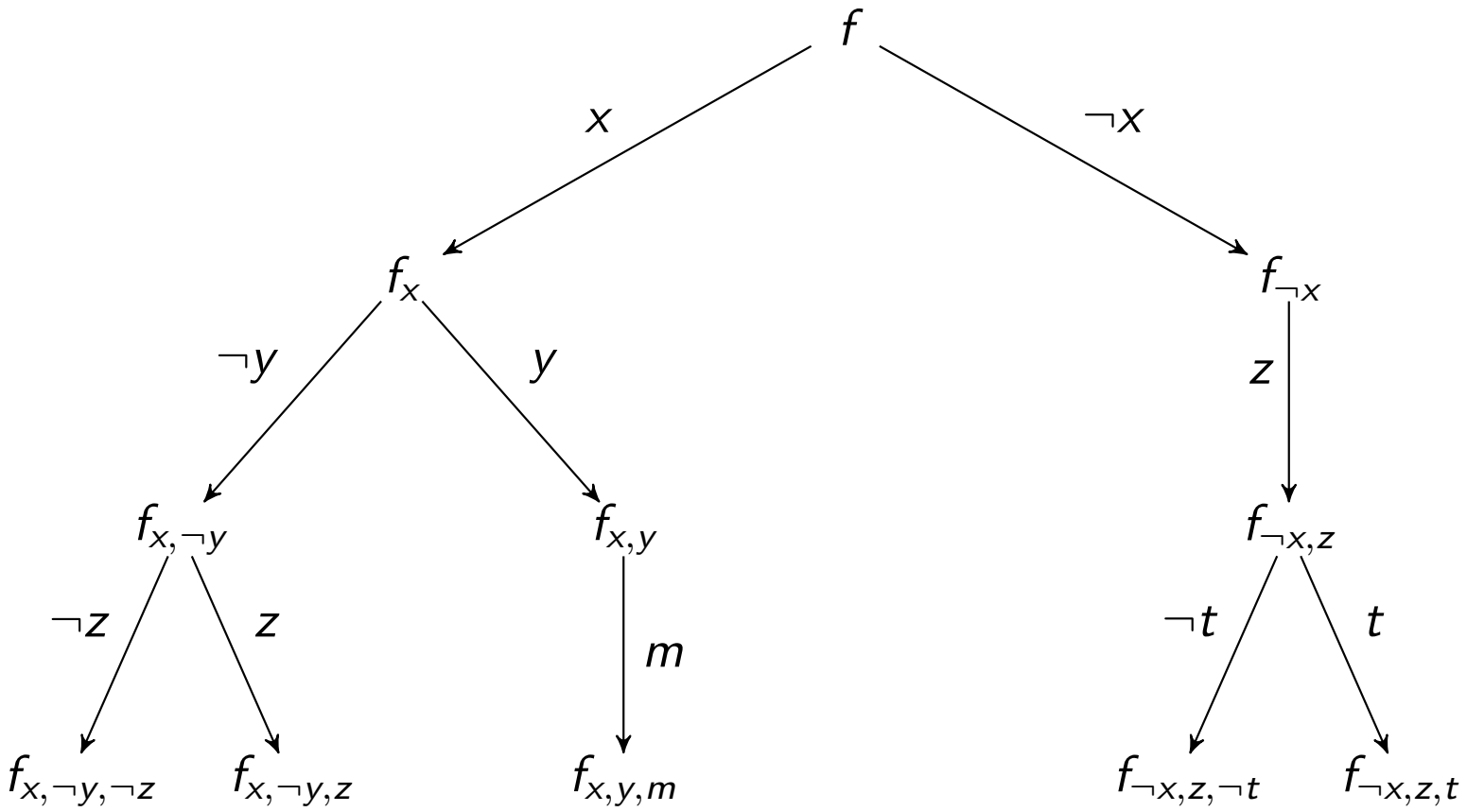
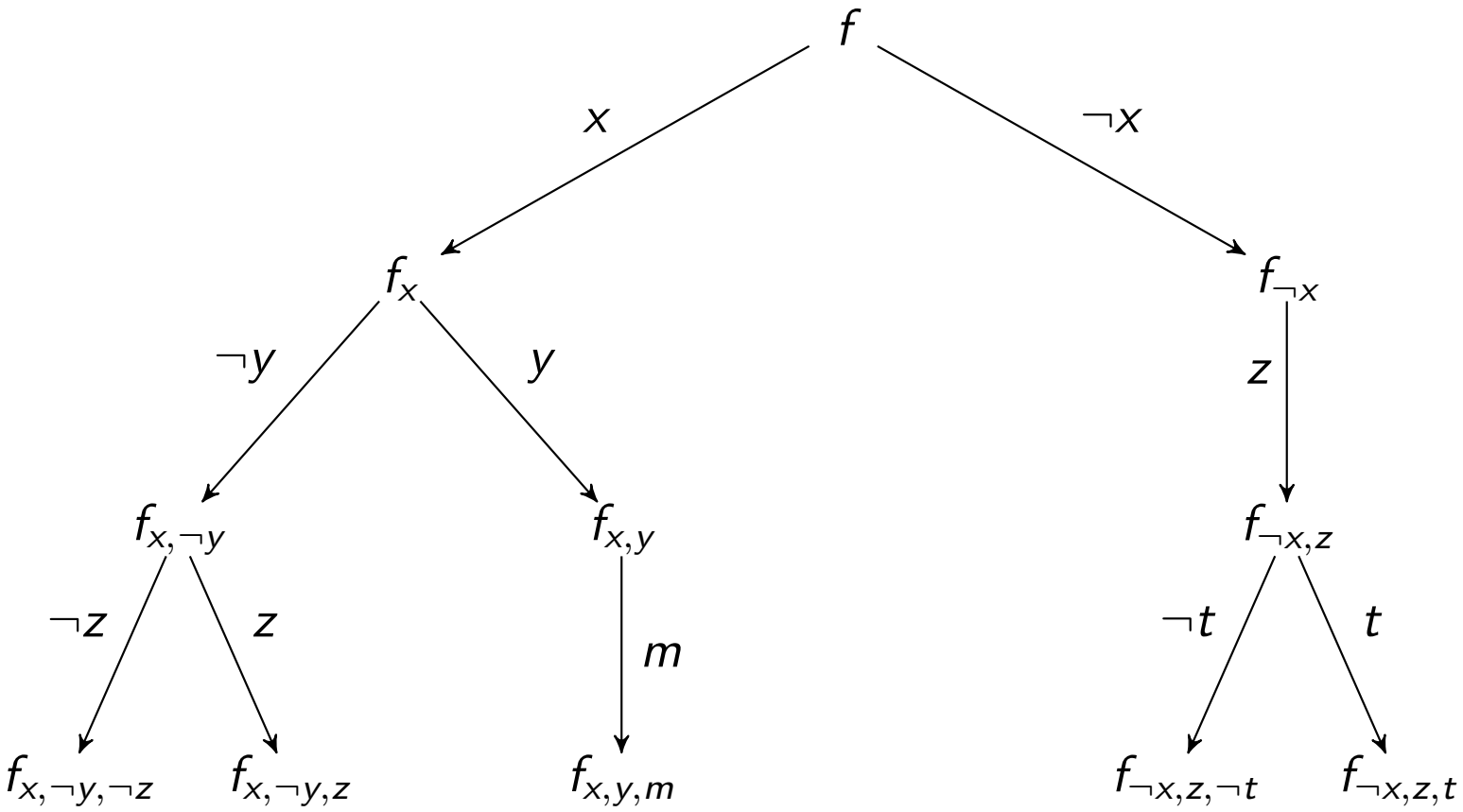
Backtrack search



- How to choose the right literal to branch on?
- First search for a model or a contradiction?

Backtrack search



- How to choose the right literal to branch on?
- First search for a model or a contradiction?

An example of DPLL

Formula	Simplified Formula	Partial Model
$x_1 \vee x_4$	$x_1 \vee x_4$	Lev. Lit. Back?
$\overline{x_1} \vee x_4 \vee x_{14}$	$\overline{x_1} \vee x_4 \vee x_{14}$	
$x_1 \vee \overline{x_3} \vee \overline{x_8}$	$x_1 \vee \overline{x_3} \vee \overline{x_8}$	
$x_1 \vee x_8 \vee x_{12}$	$x_1 \vee x_8 \vee x_{12}$	
$x_2 \vee x_{12}$	$x_2 \vee x_{12}$	
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$	$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$	
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$	$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$	
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$	$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$	

$x1$ appears in 4 clauses and 1 binary clause

An example of DPLL

Formula	Simplified Formula
$x_1 \vee x_4$	$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$	$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$	$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$	$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$	$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$	$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$	$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$	$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Partial Model

Lev.	Lit.	Back?
1	$\overline{x_1}$	(d)

x_4 appears in 1 unary clause

An example of DPLL

Formula	Simplified Formula
$x_1 \vee x_4$	$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$	$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$	$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$	$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$	$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$	$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$	$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$	$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Partial Model

Lev.	Lit.	Back?
1	$\overline{x_1}$	(d)
+	x_4	

x_3 appears in 3 clauses incl. 1 (new) binary clause

An example of DPLL

Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Simplified Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Partial Model

Lev.	Lit.	Back?
1	$\overline{x_1}$	(d)
+	x_4	
2	x_3	(d)

$\overline{x_8}$ appears in one unary clause

An example of DPLL

Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Simplified Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Partial Model

Lev.	Lit.	Back?
1	$\overline{x_1}$	(d)
+	x_4	
2	x_3	(d)
+	$\overline{x_8}$	

x_{12} appears in 1 unary clause

An example of DPLL

Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Simplified Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Partial Model

Lev.	Lit.	Back?
1	$\overline{x_1}$	(d)
+	x_4	
2	x_3	(d)
+	$\overline{x_8}$	
+	x_{12}	

x_{13}, $\overline{x_7}$ appear in unary clauses

An example of DPLL

Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Simplified Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Partial Model

Lev.	Lit.	Back?
1	$\overline{x_1}$	(d)
+	x_4	
2	x_3	(d)
+	$\overline{x_8}$	
+	x_{12}	
+	x_{13}	

x_7, $\overline{x_7}$ appear in unary clauses

An example of DPLL

Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$
$\overline{x_3} \vee \overline{x_{12}} \vee x_{13}$
$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Simplified Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
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$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Partial Model

Lev.	Lit.	Back?
1	$\overline{x_1}$	(d)
+	x_4	
2	x_3	(d)
+	$\overline{x_8}$	
+	x_{12}	
+	x_{13}	
+	$\overline{x_7}$	

Conflict! Undo everything until last decision

An example of DPLL

Formula

$x_1 \vee x_4$
$\overline{x_1} \vee x_4 \vee x_{14}$
$x_1 \vee \overline{x_3} \vee \overline{x_8}$
$x_1 \vee x_8 \vee x_{12}$
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Simplified Formula

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$\overline{x_3} \vee x_7 \vee \overline{x_{13}}$
$x_8 \vee \overline{x_7} \vee \overline{x_{12}}$

Partial Model

Lev.	Lit.	Back?
1	$\overline{x_1}$	(d)
+	x_4	
*	$\overline{x_3}$	

Now, $\overline{x_3}$ is not a decision

A very simple procedure?

Very simple to write a backtrack search but...

Where to branch?

- **Mistakes at the top of the tree are dramatic!**
- **(almost) As many nodes where to decide than where to explore**
- **A perfect branching scheme is NP-Hard**

Today's Itinerary

Bounded Model Checking at a glance

We have a system to verify, modeled by an automaton, encoding its state transitions

Correctness: No bugs A special state "error" is used in the model. The problem is about its reachability.

Liveness: No infinite loop Any state must be reachable from any other state, in any future.

Notice:

- Closely related to temporal logic;
- Before SAT, BDD were used to solve these problems

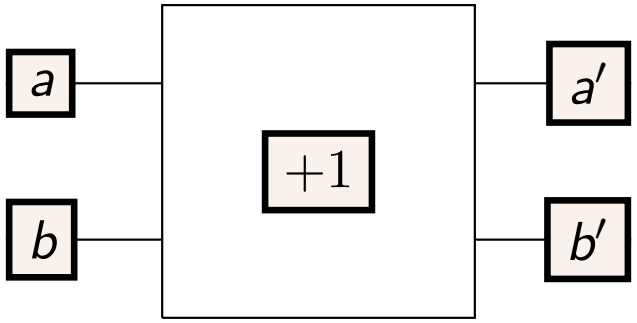
(Bounded) Model Checking at a glance

We fix a *bound* k, and increment it as we need

- The automaton is represented by the propositional logic function T that encodes the characteristics function of the reachable states.

Example (2-bits 1-adder) :

$$(a' \leftrightarrow \neg a) \wedge (b' \leftrightarrow a \oplus b)$$



$$(0,0) \rightarrow (1,0) \rightarrow (0,1) \rightarrow (1,1) \rightarrow (0,0) \rightarrow \ldots$$

(Bounded) Model Checking at a glance

We fix a *bound* k, and increment it as we need

- The automaton is represented by the propositional logic function T that encodes the characteristics function of the reachable states. Example (2-bits 1-adder) : $(a' \leftrightarrow \neg a) \wedge (b' \leftrightarrow a \oplus b)$
- The property to check is (for instance): $a \wedge b$ (is the state (11) reachable?)
- The initial state is an assignment of variables at time step 0

BMC: Unrolling loops

Let us check whether the state (11) is reachable in 2 iterations

$I(s_0) = \neg a_0 \wedge \neg b_0$

$T(s_0, s_1) = (a_1 \leftrightarrow \neg a_0) \wedge (b_1 \leftrightarrow a_0 \oplus b_0)$

$T(s_1, s_2) = (a_2 \leftrightarrow \neg a_1) \wedge (b_2 \leftrightarrow a_1 \oplus b_1)$

$p(s_2) = a_2 \wedge b_2$

$p(s_0) = a_0 \wedge b_0$

$p(s_1) = a_1 \wedge b_1$

Finally, is the formula

$(\neg a_0 \wedge \neg b_0) \wedge ((a_1 \leftrightarrow \neg a_0) \wedge (b_1 \leftrightarrow a_0 \oplus b_0)) \wedge ((a_2 \leftrightarrow \neg a_1)$
$\wedge (b_2 \leftrightarrow a_1 \oplus b_1)) \wedge (a_2 \wedge b_2)$ satisfiable?

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Let us check whether the state (11) is reachable in 2 iterations

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$T(s_0, s_1) = (a_1 \leftrightarrow \neg a_0) \wedge (b_1 \leftrightarrow a_0 \oplus b_0)$

$T(s_1, s_2) = (a_2 \leftrightarrow \neg a_1) \wedge (b_2 \leftrightarrow a_1 \oplus b_1)$

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$p(s_0) = a_0 \wedge b_0$

$p(s_1) = a_1 \wedge b_1$

Finally, is the formula

$(\neg a_0 \wedge \neg b_0) \wedge ((a_1 \leftrightarrow \neg a_0) \wedge (b_1 \leftrightarrow a_0 \oplus b_0)) \wedge ((a_2 \leftrightarrow \neg a_1)$
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$T(s_0, s_1) = (a_1 \leftrightarrow \neg a_0) \wedge (b_1 \leftrightarrow a_0 \oplus b_0)$

$T(s_1, s_2) = (a_2 \leftrightarrow \neg a_1) \wedge (b_2 \leftrightarrow a_1 \oplus b_1)$

$p(s_2) = a_2 \wedge b_2$

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$p(s_1) = a_1 \wedge b_1$

Finally, is the formula

$(\neg a_0 \wedge \neg b_0) \wedge ((a_1 \leftrightarrow \neg a_0) \wedge (b_1 \leftrightarrow a_0 \oplus b_0)) \wedge ((a_2 \leftrightarrow \neg a_1)$
$\wedge (b_2 \leftrightarrow a_1 \oplus b_1)) \wedge (a_2 \wedge b_2)$ satisfiable?

Unbounded Model Checking

$$I \wedge T_1 \wedge T_2 \wedge \ldots \wedge T_k \wedge BUG_k$$

How to ensure that BUG is unreachable?

Idea : find an invariant Inv s.t. BUG is not reachable in $k > 0$ steps

- Inv characterizes an over approximation of the reachable states in j steps:

$$I \wedge T_1 \wedge \cdots \wedge T_j \rightarrow Inv$$

- Inv is an inductive property:

$$Inv \wedge T_1 \rightarrow Inv_1$$

- BUG is not reachable from Inv in k steps:

$$Inv \wedge T_1 \wedge T_2 \wedge \ldots \wedge T_k \wedge BUG_k \equiv \bot$$

- Incremental SAT Solving / Proof Analysis

The Erdös Discrepancy Problem (1932)

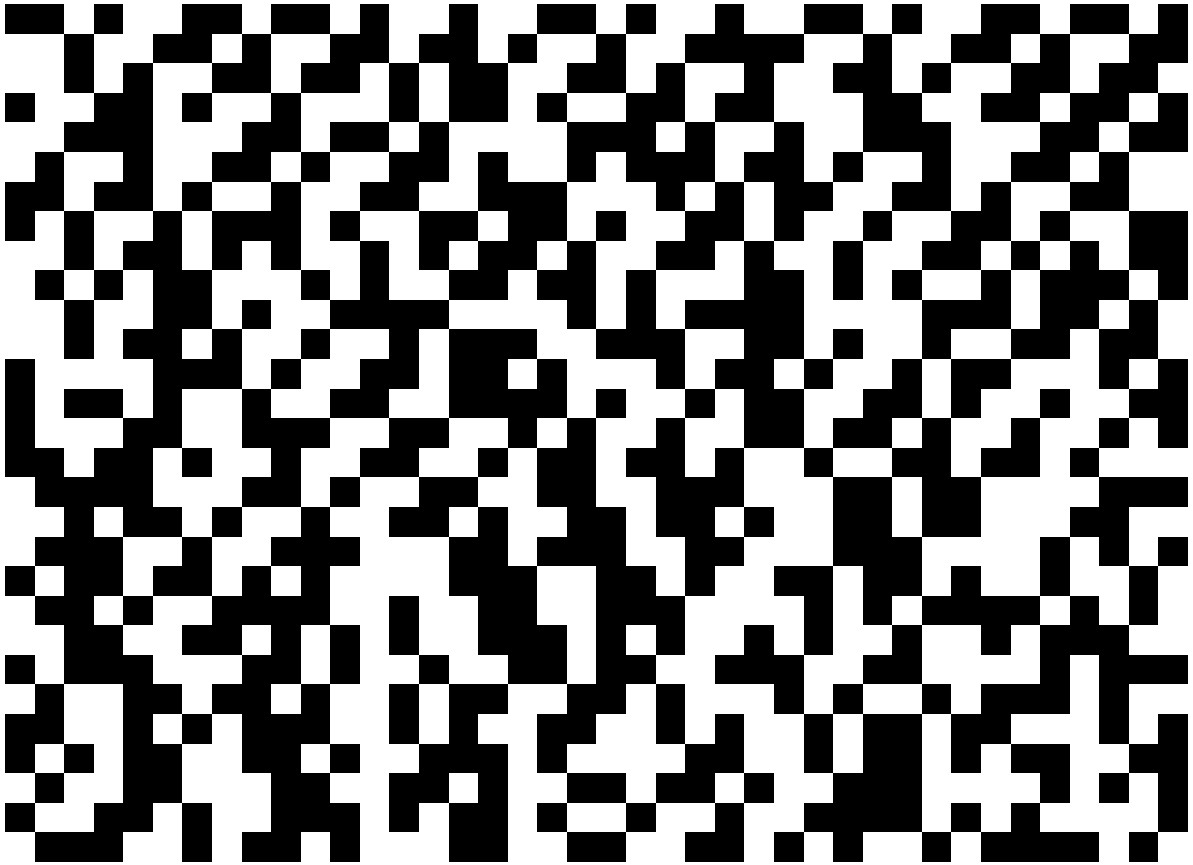
- Infinite series of +1 and -1: $< -1, 1, 1, -1 - 1, 1, 1, \ldots >$
- $\forall C\ \exists k, d\ t.q.\ \ |\sum_{i=1}^{k} x_{i.d}| \geq C$

$$
\begin{array}{ccccccccccc|c}
+ & - & - & + & + & - & - & - & + & + & - & -1 \\
 & - & & + & & - & & - & & + & & -1 \\
 & & - & & & - & & & + & & & -1 \\
 & & & + & & & & - & & & & 0 \\
 & & & & + & & & & & + & & +2
\end{array}
$$

- Proven in 2014 for $C = 2$ (k=1161)
- The *proof*: UNSAT certificate (trace) from Glucose (13 Gb)[3]
- General case proven two years later by Terence Tao (previous proof considered as the biggest mathematical proof ever by T. Tao).

[3] `cgi.csc.liv.ac.uk/~konev/SAT14/`

Solution for C=2, 1160 steps

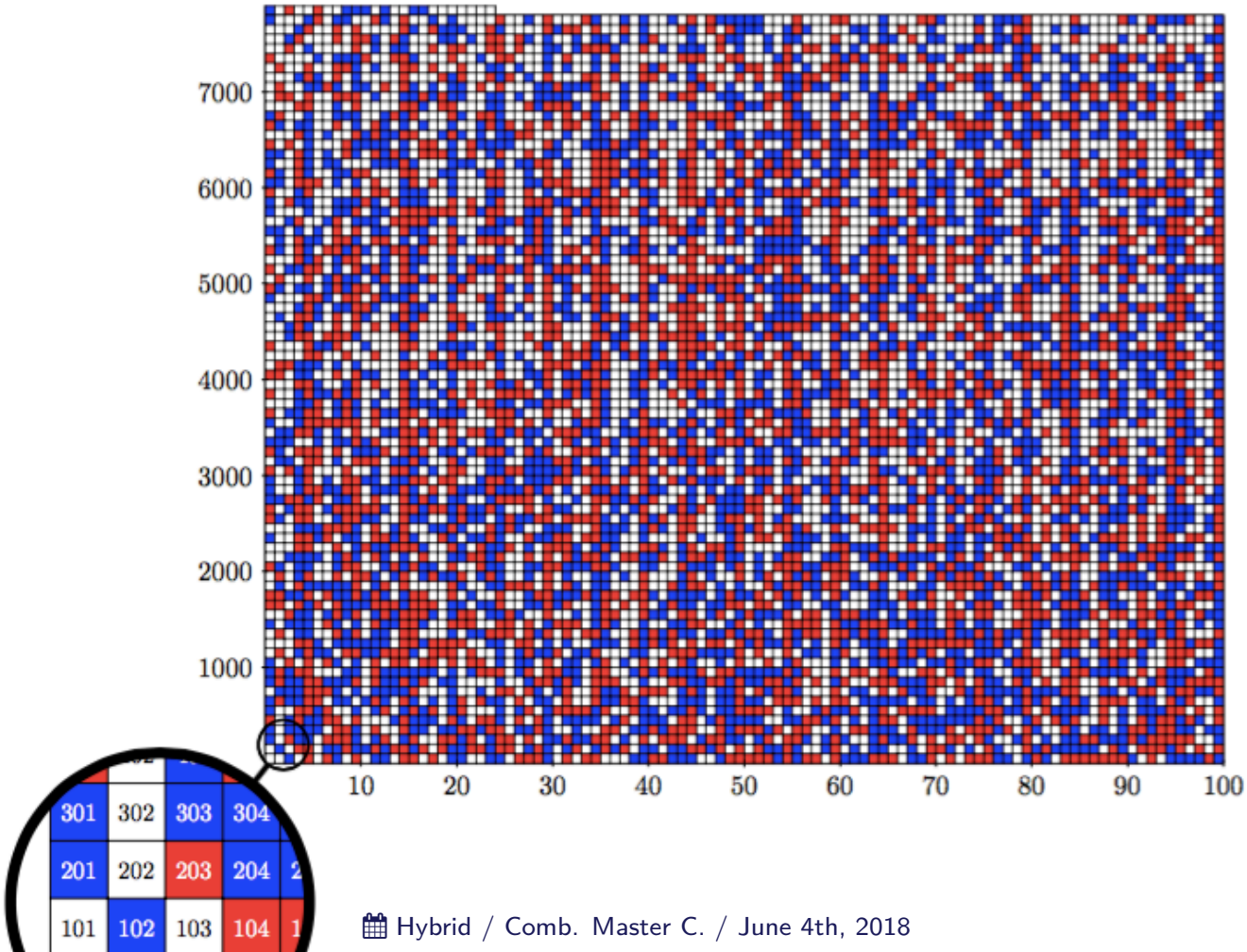


(For C=3, maximum solution is not yet known)

The "biggest proof" in the world

Boolean Pythagorean triples problem

Is it possible to colorize the n integers $\leq n$ in two colors s.t. no triplet (a, b, c) is $a^2 + b^2 = c^2$ monochromatic?



The "biggest proof" in the world

No solution for n=7825

- Open question since 20 years
- 10^{2300} possible canidates
- SAT encoding
- Original problem splitted in 1,000,000 subproblems
- 800 CPUs

Proof is 200Tb long (Glucose's output)

- In practice the proof is not really kept

Let's generate this 200Tb proof! (or not)

Follow the `PythagoreanTriplets` Python Notebook!

Today's Itinerary

From LookAhead to Lookback

All solvers are now turned to lazily detect Unit Propagation

No way to maintain counters for "smart" branching

Look ahead heuristics were "easy" to understand

Look back heuristics are very hard to study

1999, on the way to the revolution

Huge problems are coming from the real-world: Planning & Bounded Model Checking

- Planning as Satisfiability. [Kautz and Selman, 92]
- Symbolic Model Checking using SAT procedures instead of BDDs. [Biere & al. 99]
- SAT solvers can't cope with those huge formulas without specialized data structures Quasi-incapacité des solveurs SAT à gérer autant de clauses

DPLL extinction...

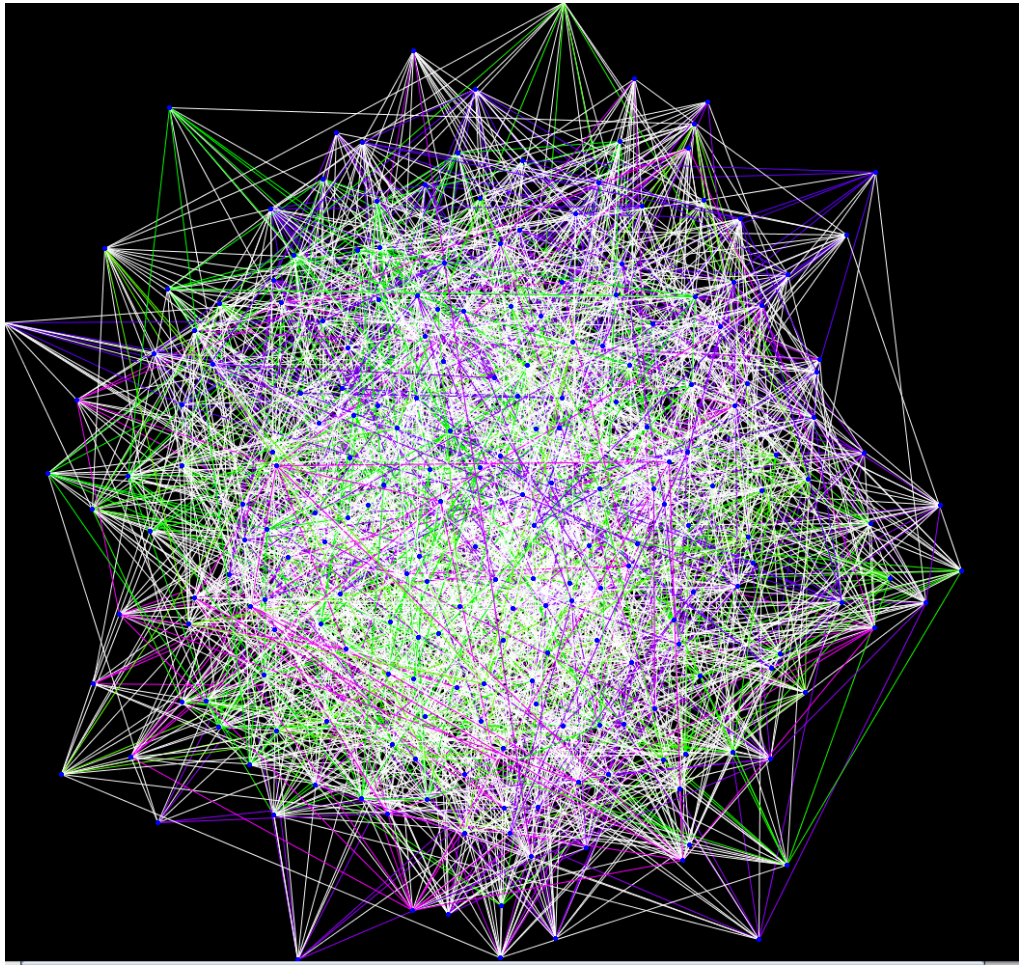
- GRASP : **learning clauses** in SAT solvers
- DLIS : **very simple heuristic**
- SATO : **lazy data structure** to detect unary clauses

Algorithms ingredients for the upcoming revolution
BMC, GRASP, DLIS, SATO

Huge problems?

Generated by the tool : satgraf http://satbench.uwaterloo.ca/site/satgraf

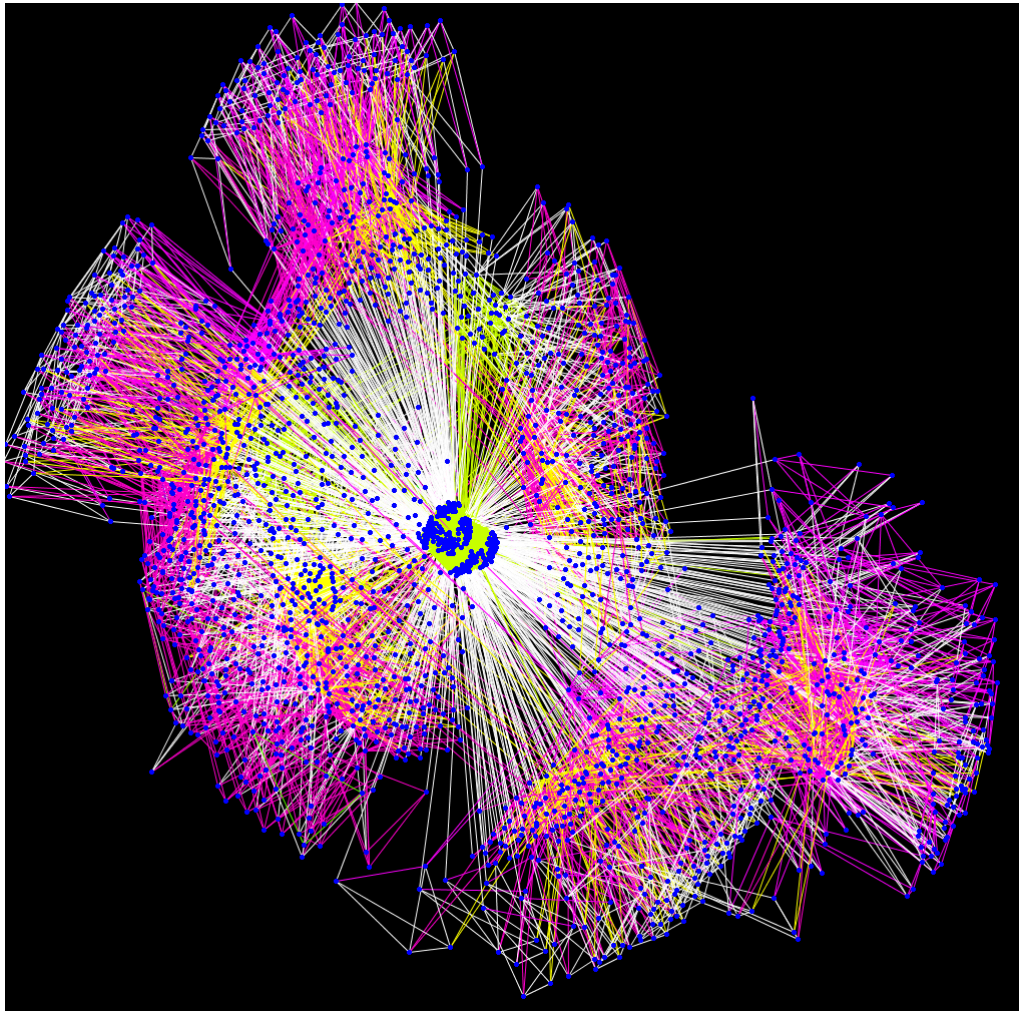
Random formula



Huge problems?

Generated by the tool : satgraf http://satbench.uwaterloo.ca/site/satgraf

BMC instances



Huge problems?

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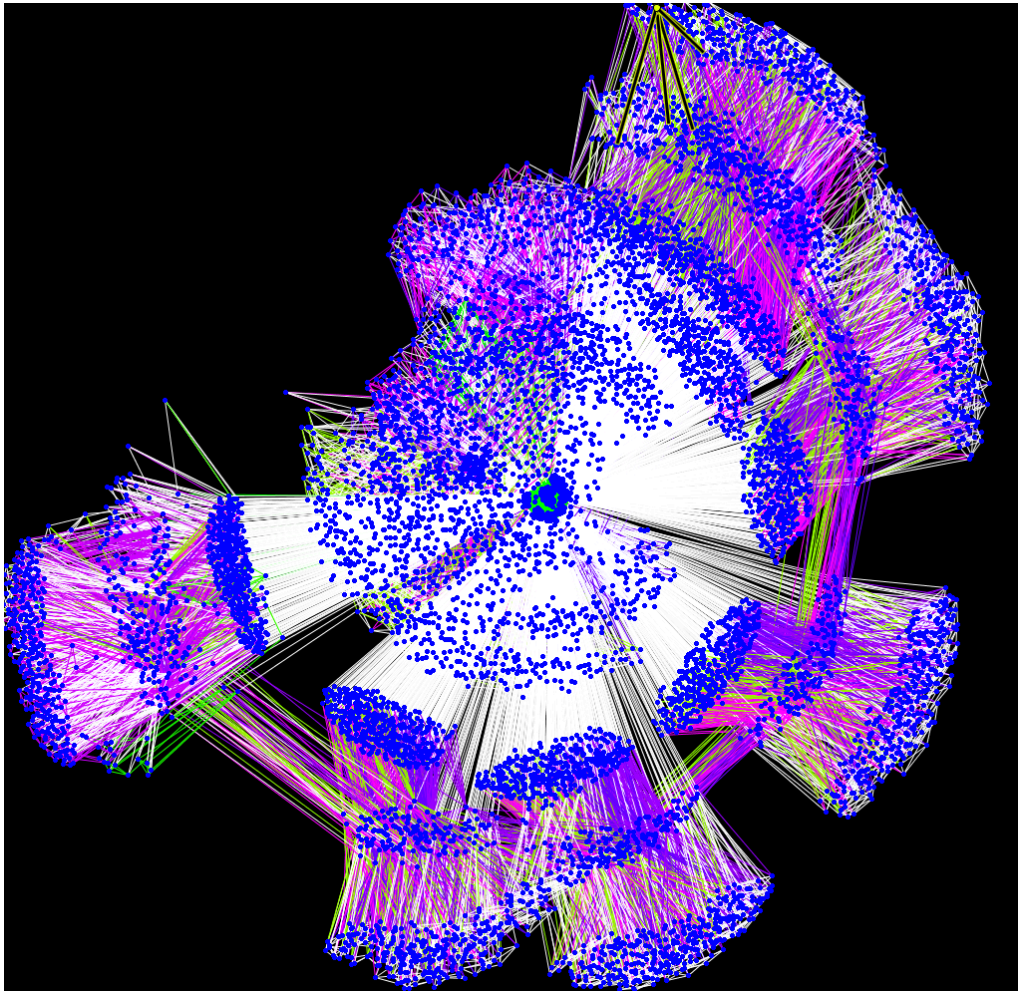
BMC instances



Huge problems?

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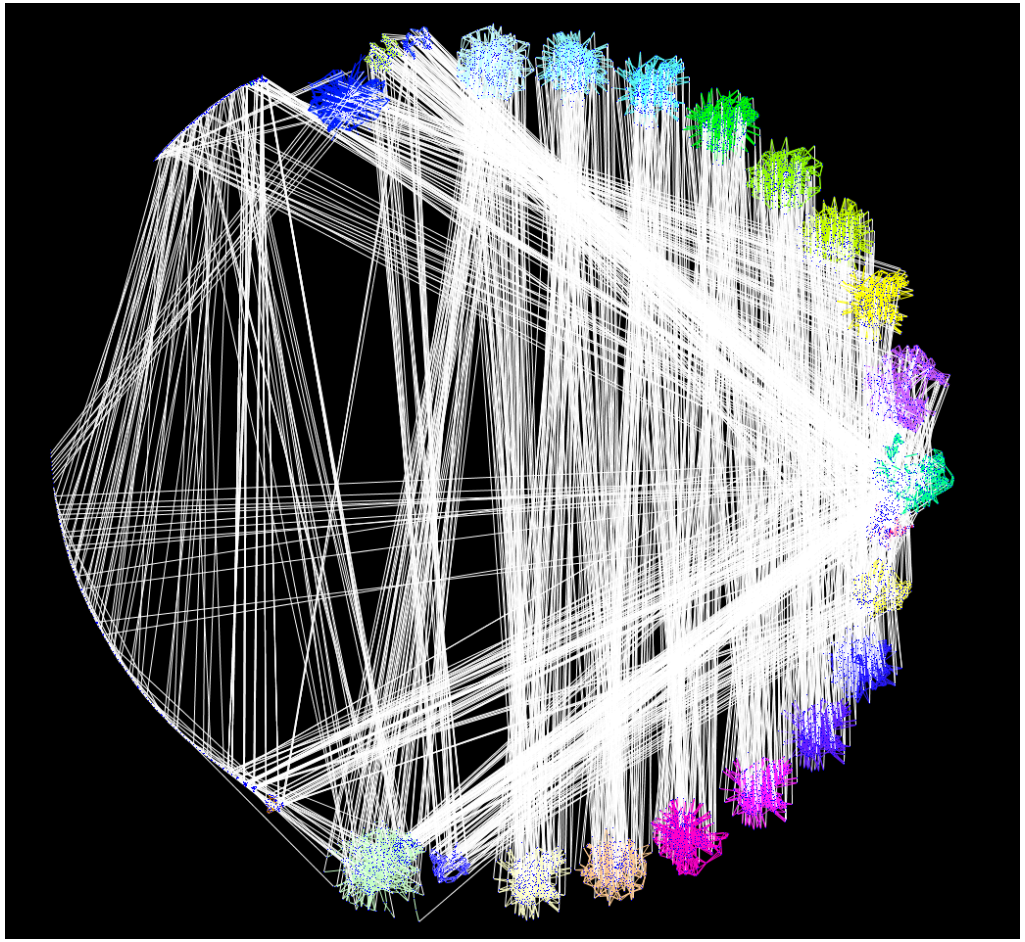
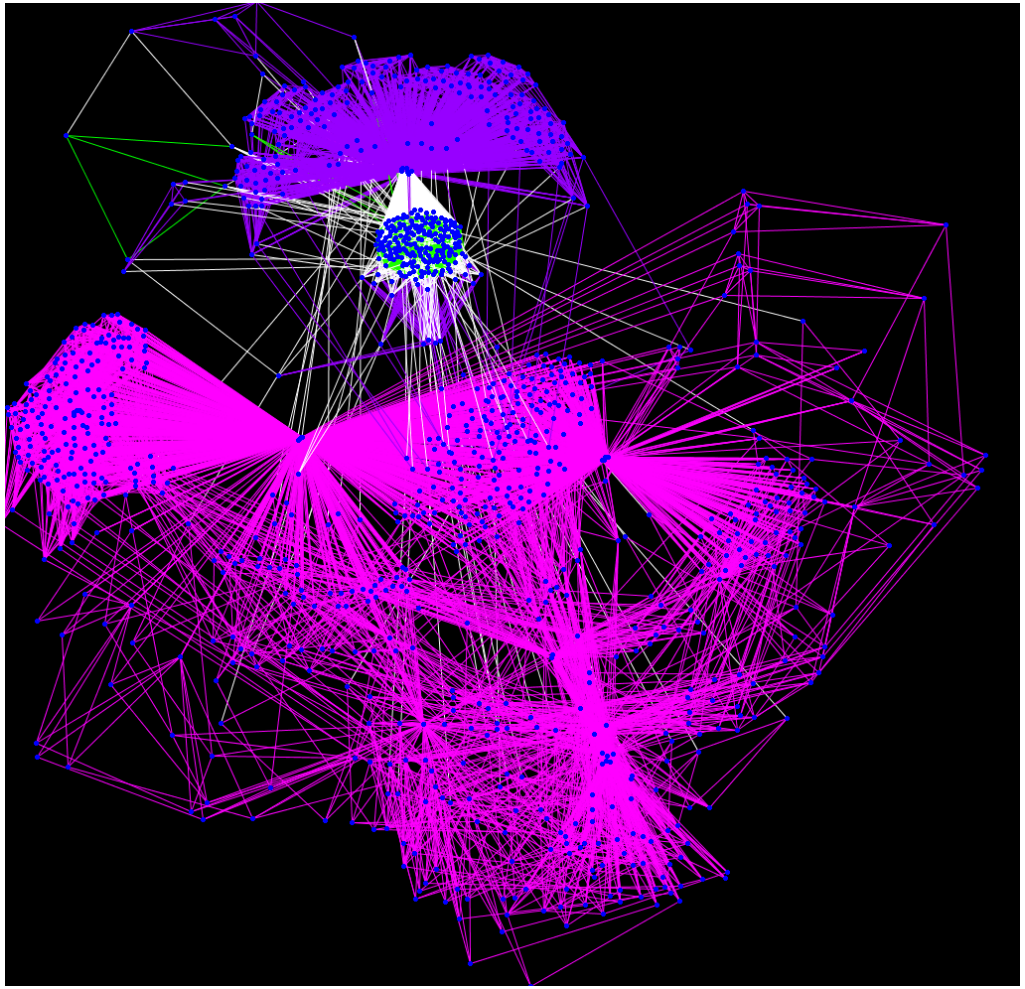
BMC instances



Huge problems?

Generated by the tool : satgraf http://satbench.uwaterloo.ca/site/satgraf

BMC instances



Data structures are stronger than algorithms!

Until now:

- Heuristics were used to *mimic* what a human will do (when picking a variable to branch on)
- Data structures were implemented to support algorithms

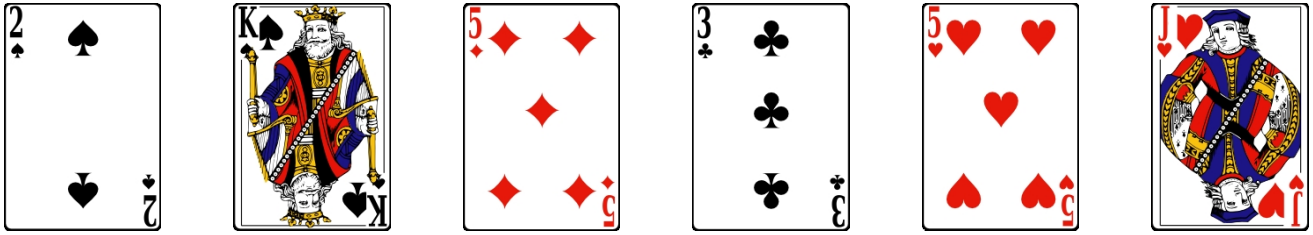
Seminal paper
« *Chaff: Engineering an Efficient SAT Solver* » [Moskewicz & al. '01]
"Simply" optimize known algorithms

Highest priority: BCP (Boolean Constraint Propagation)

Let's play...to be the laziest

Detect when you have only 1 non called card in your hand

1. **count the non called cards**



6

Let's play...to be the laziest

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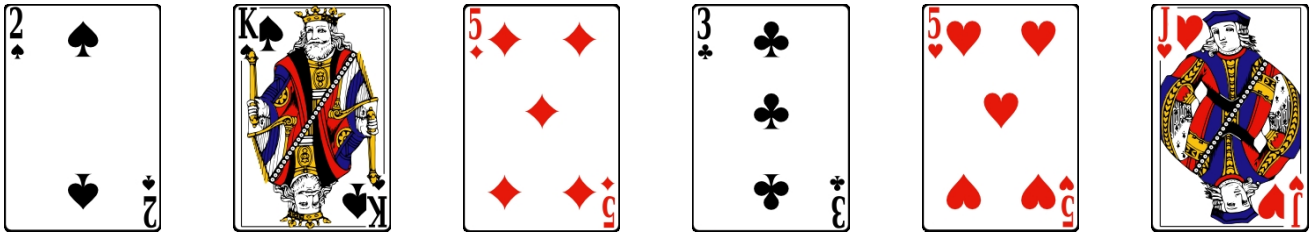
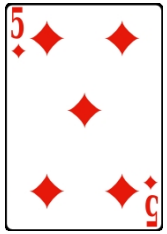


6



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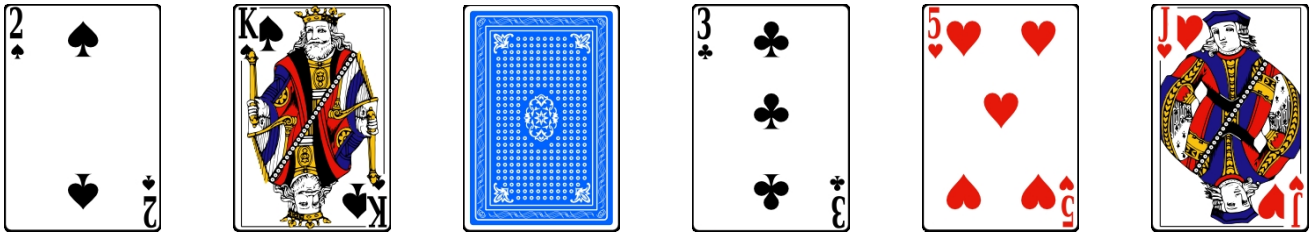

5

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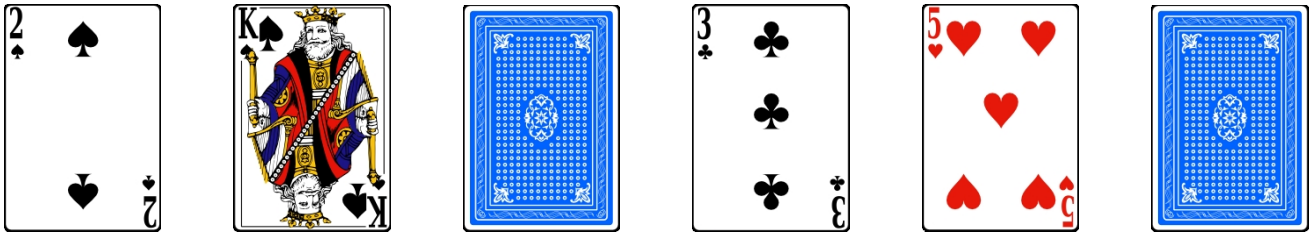

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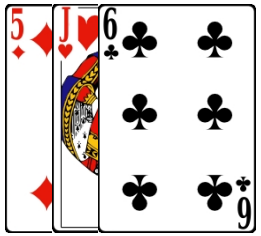

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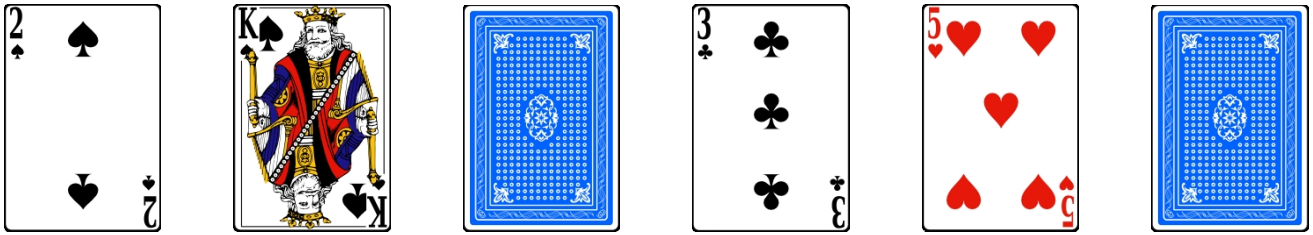

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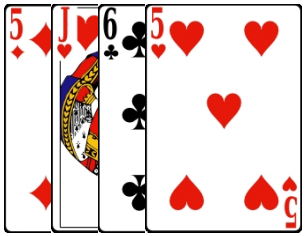

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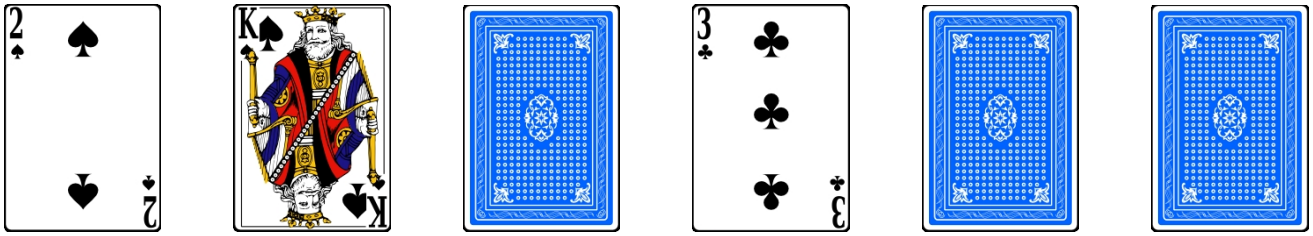

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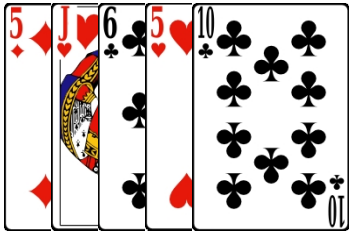

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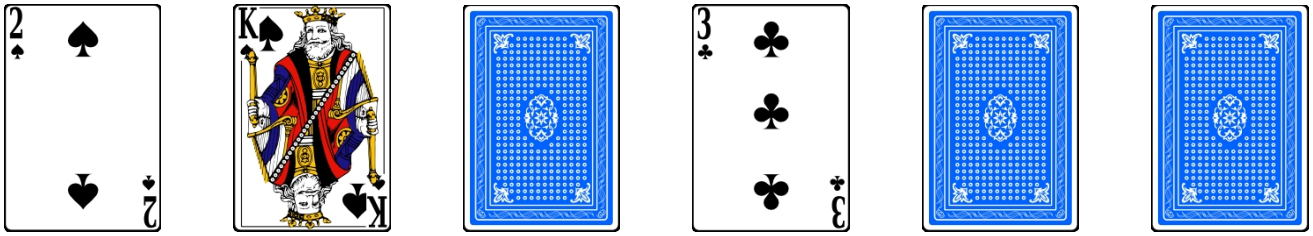

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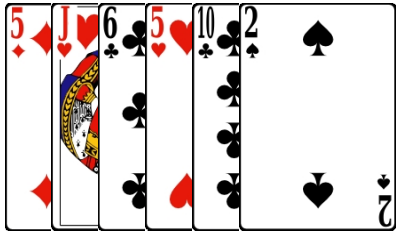

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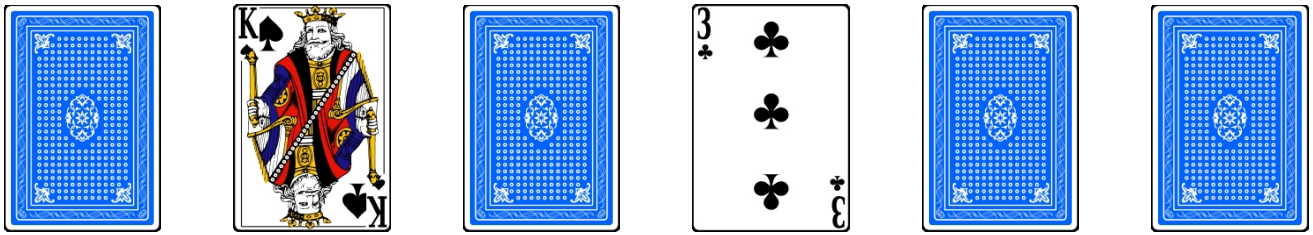

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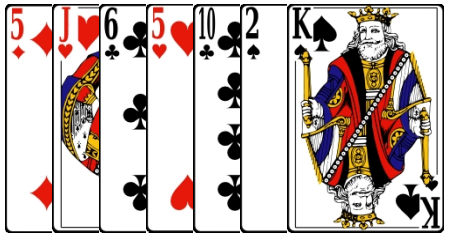

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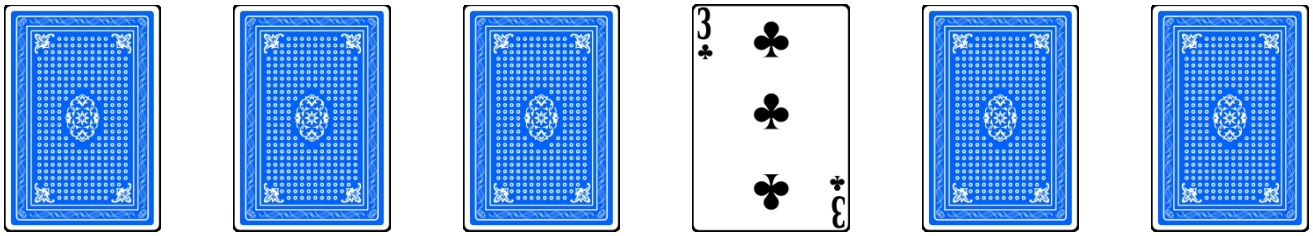
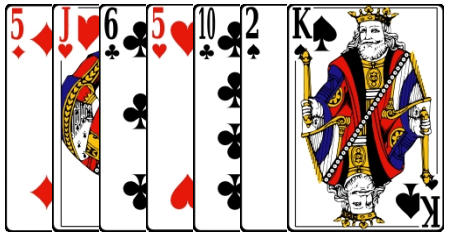


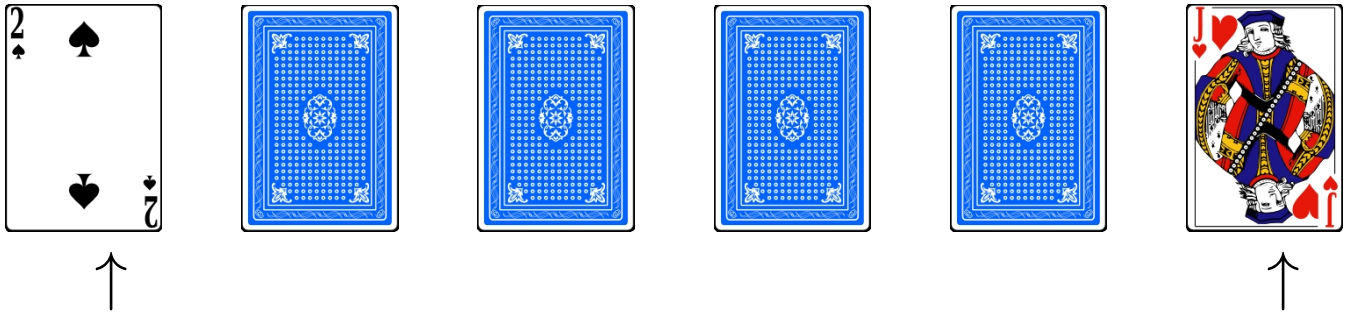
Unary Clause!!



Let's play...to be the laziest – 2

Detect when you have only 1 non called card in your hand

1. **Watch heads and tails**



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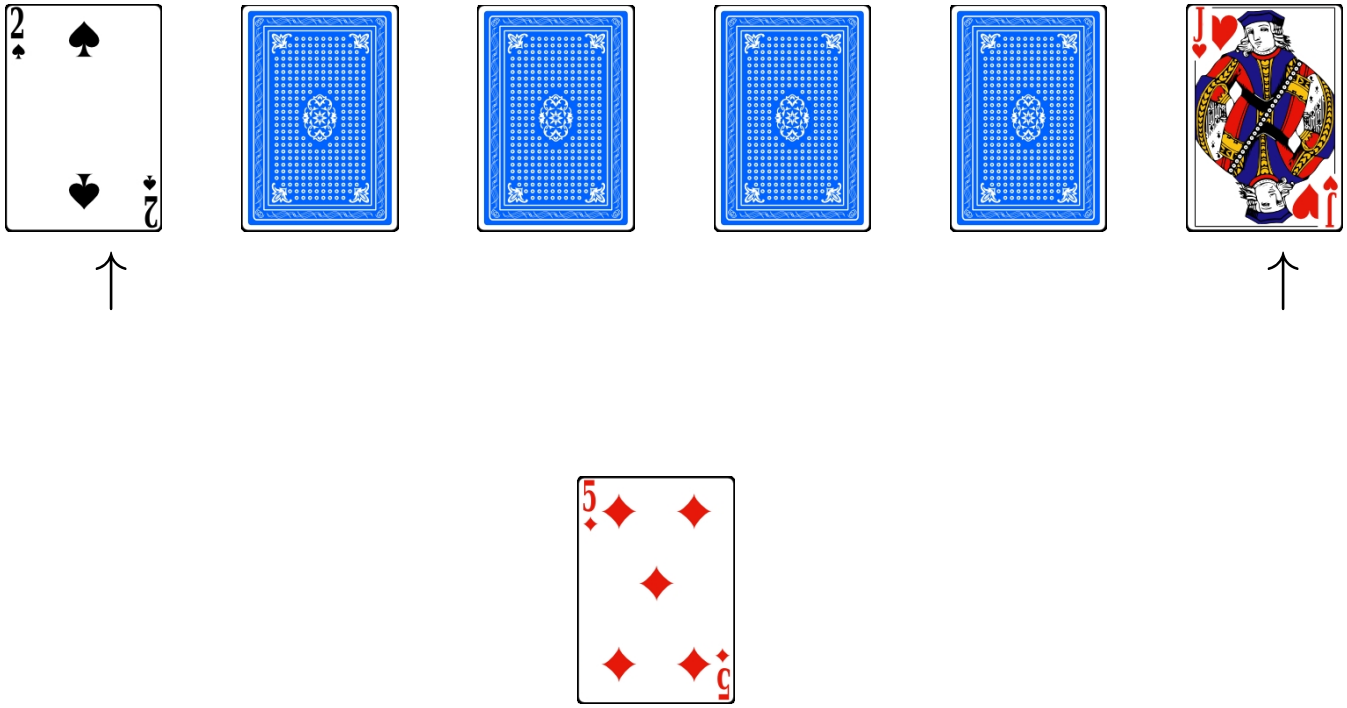

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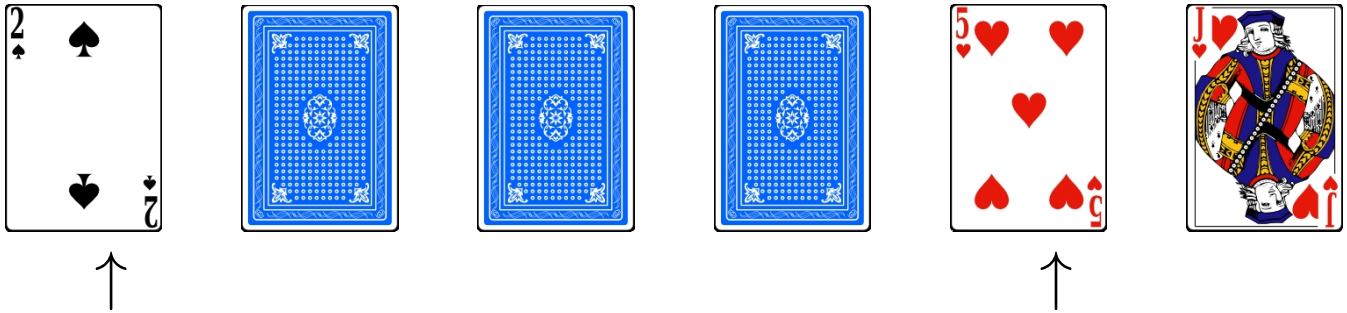

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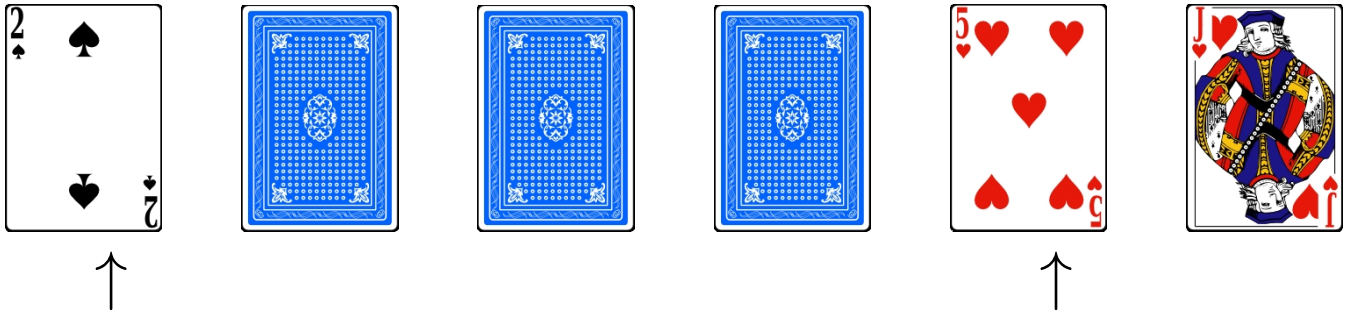

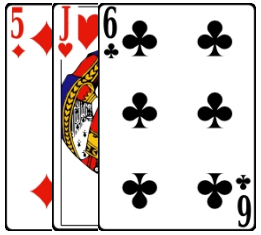

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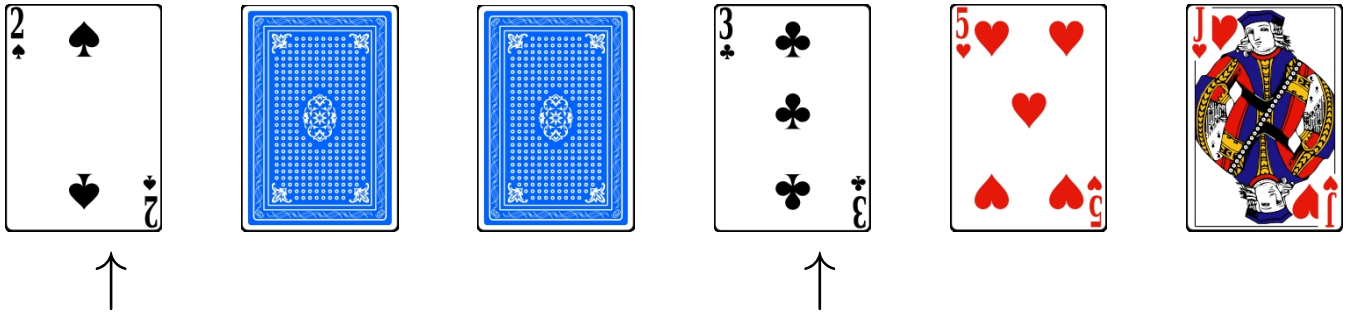

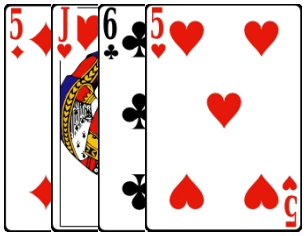

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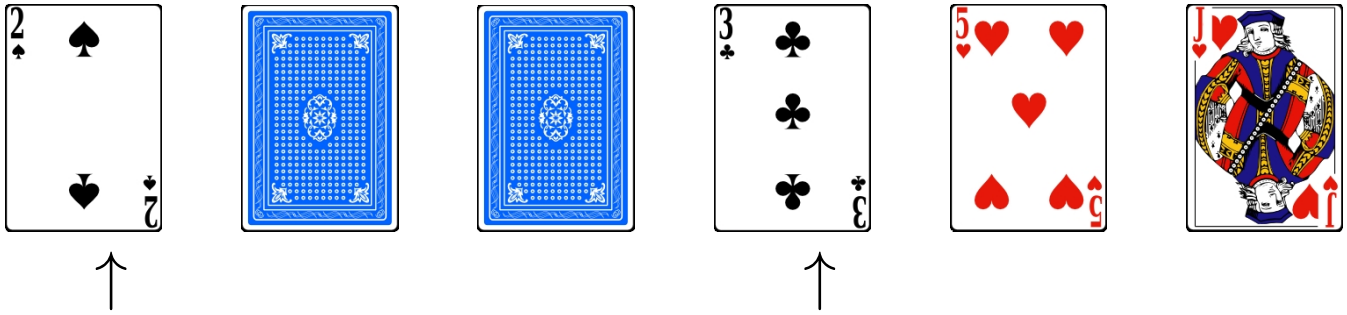

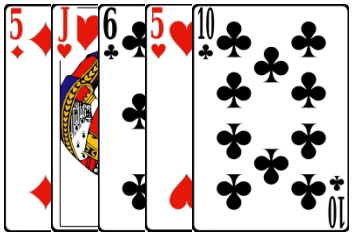

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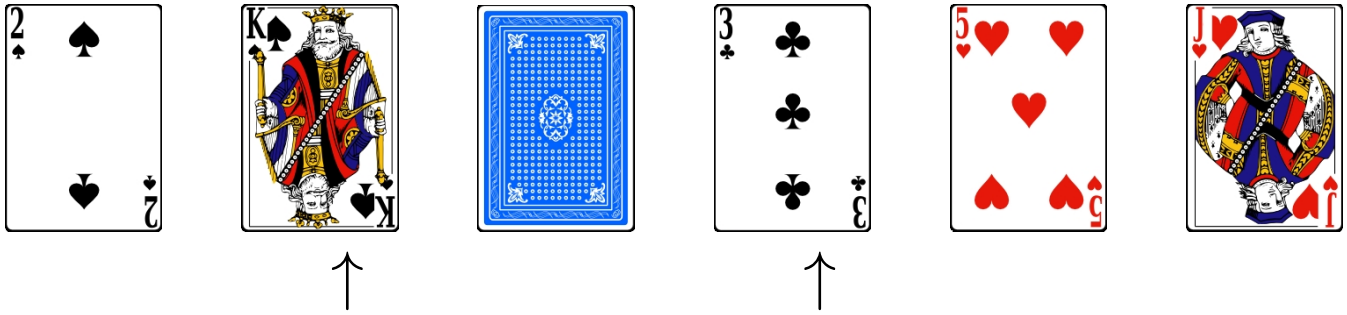

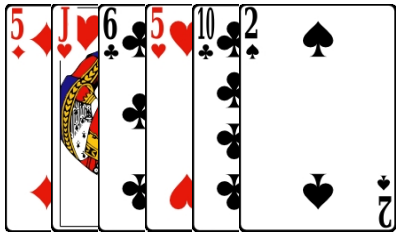

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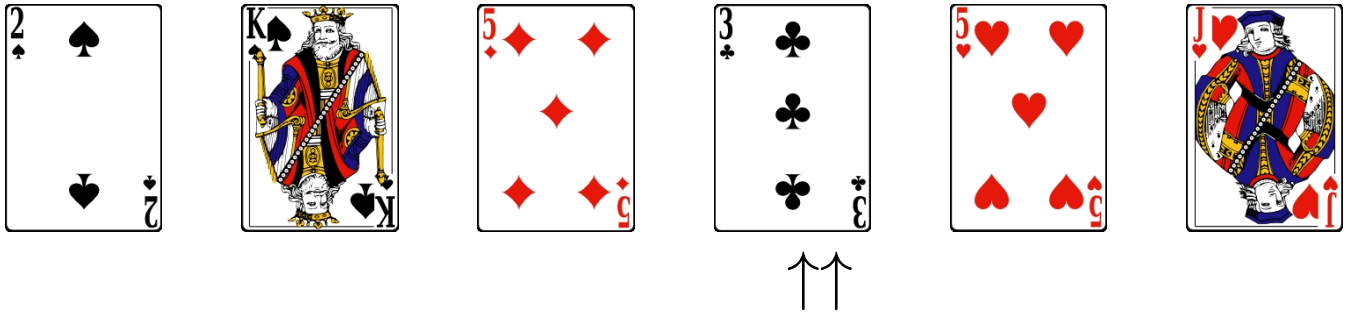

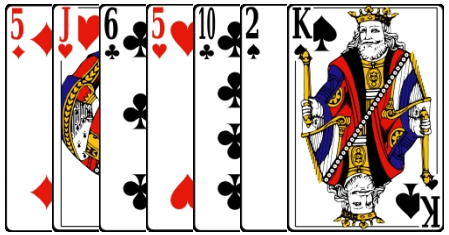

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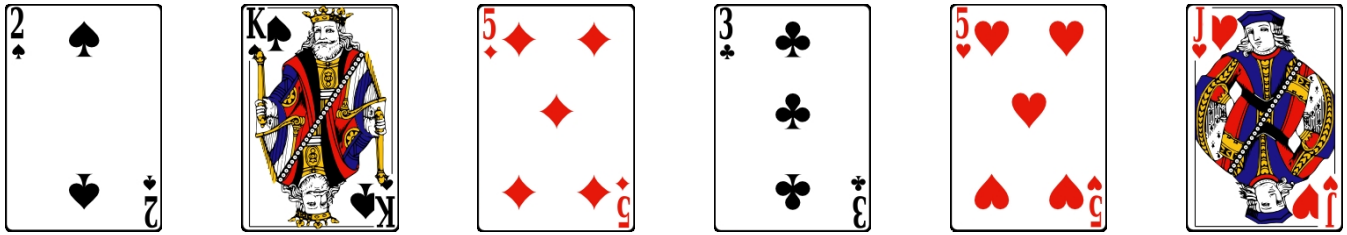

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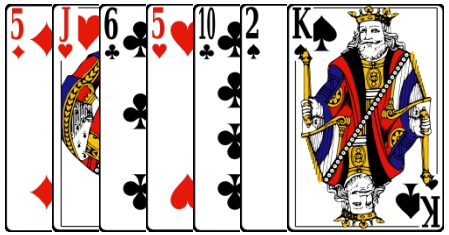

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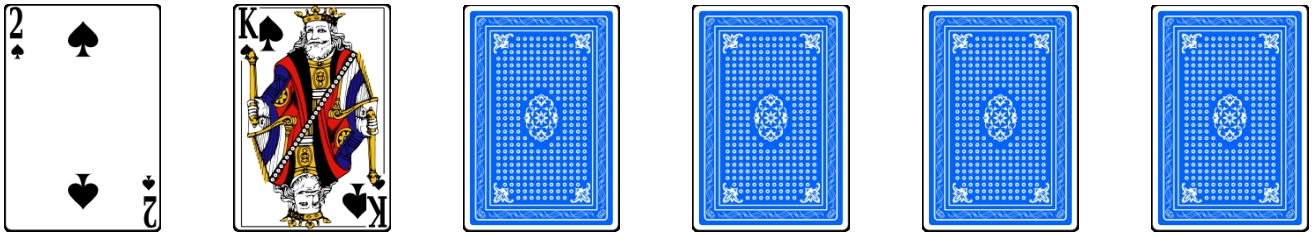

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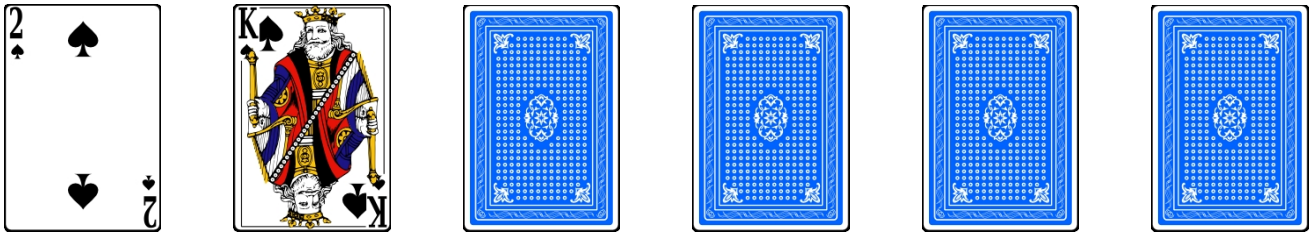

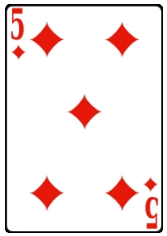

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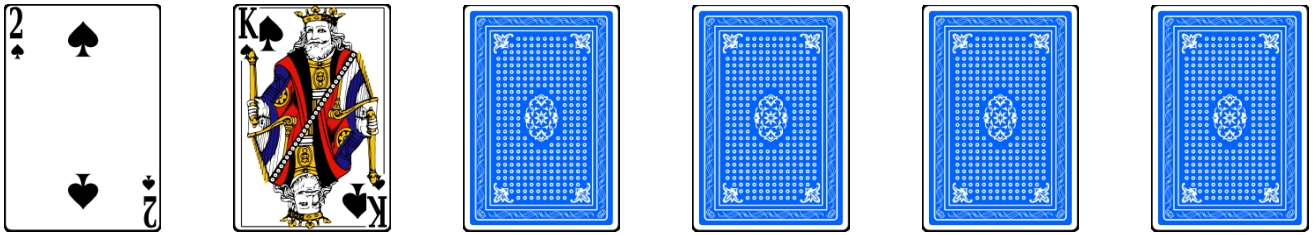

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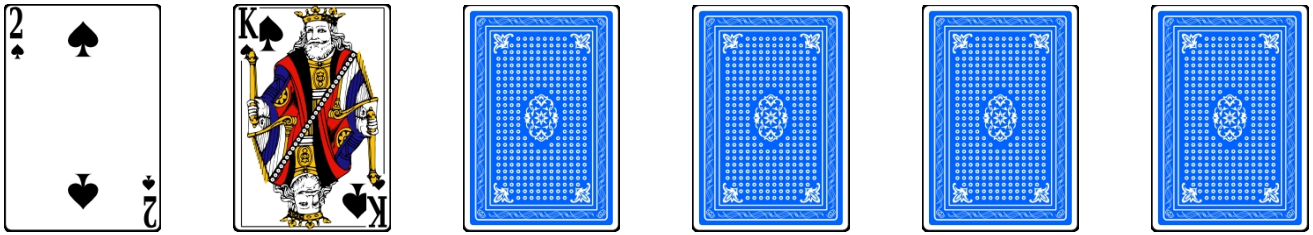

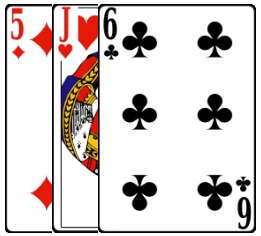

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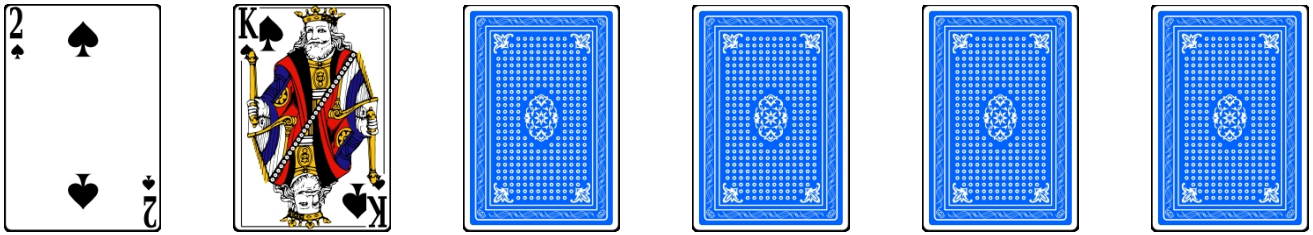

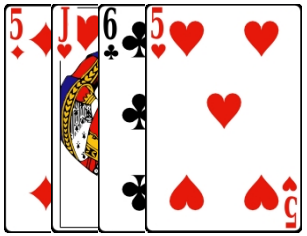

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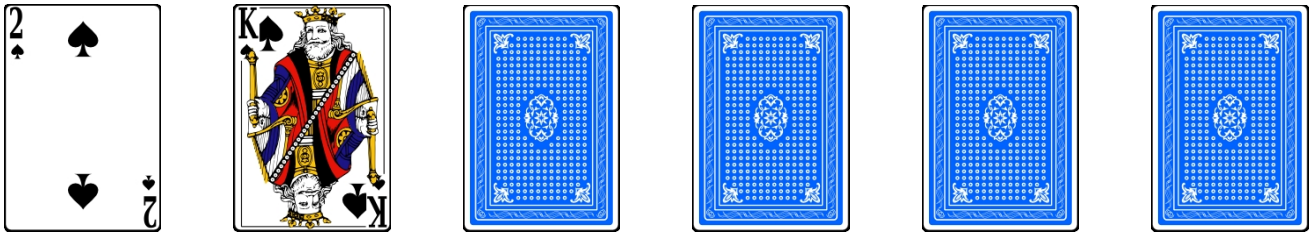

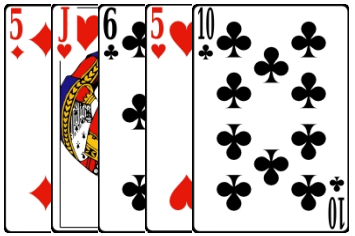

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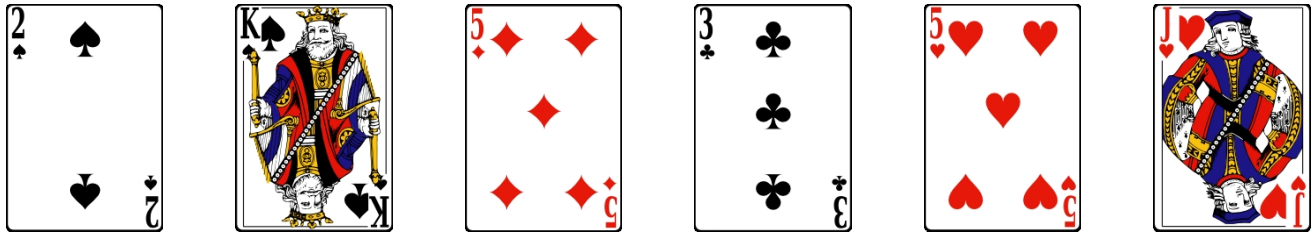

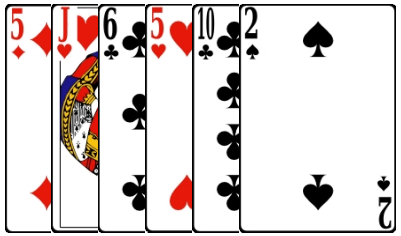

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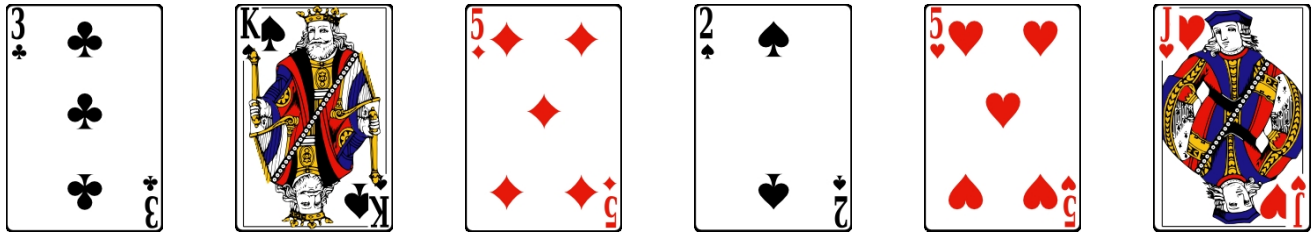

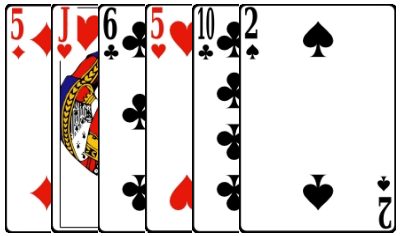

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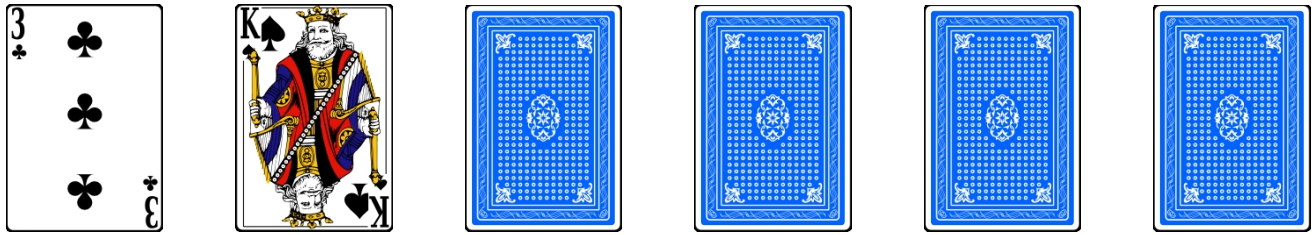

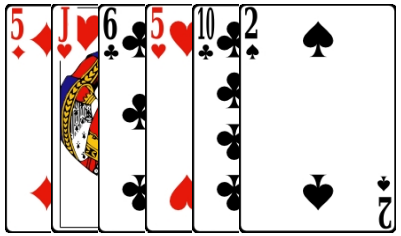

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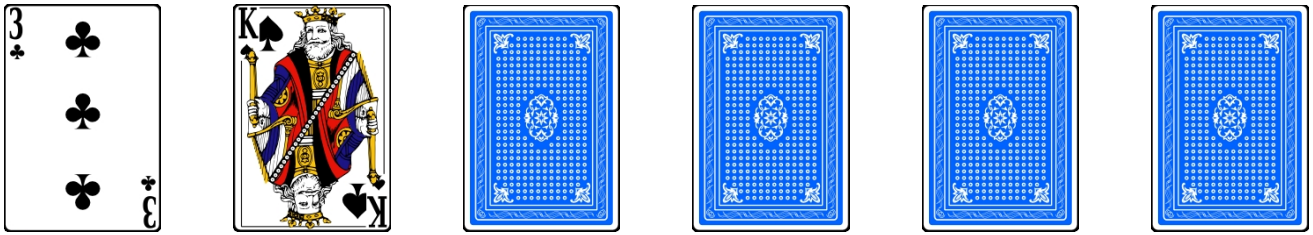

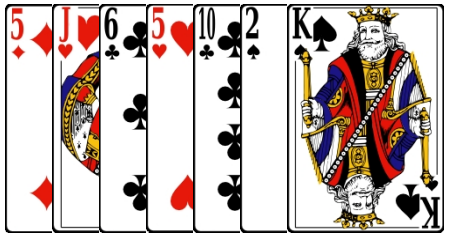

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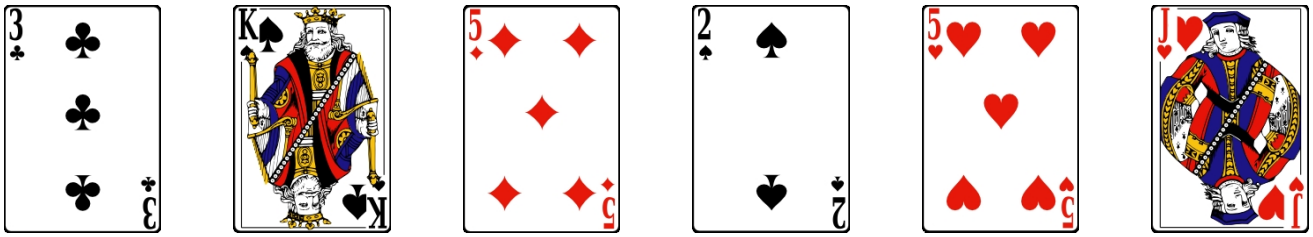

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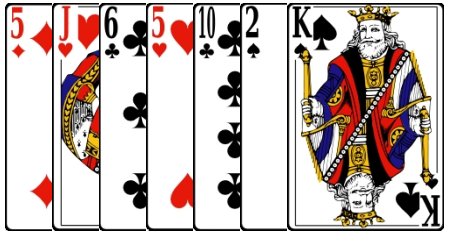

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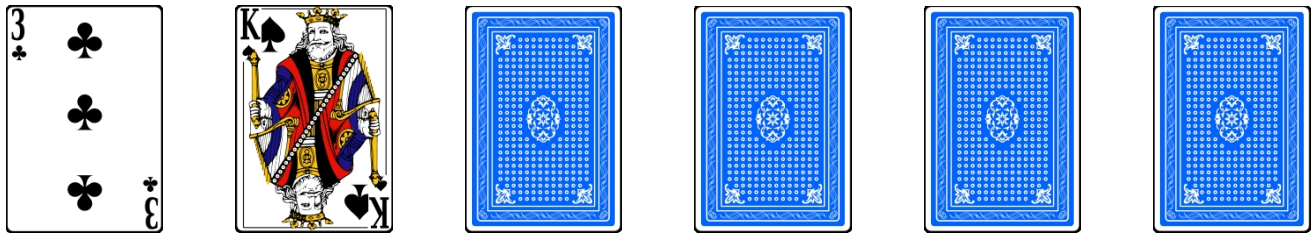
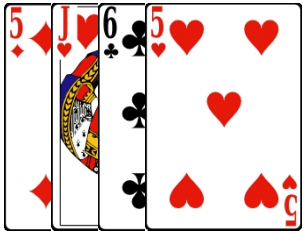


Nothing to do if want to replay!!



Only 2 watches per clauses

How to update it?

Let's say the current assignement is a, b, c, d

We branch on e

Clauses that can be unary/empty are all the clauses watched by $\neg e$

$\neg e \vee a \vee f \vee g$	Watched by $\neg e, a$
$\neg e \vee f \vee b$	Watched by $\neg e, f$
$f \vee \neg e \vee \neg a \vee \neg g$	Watched by $\neg e, f$
$\neg e \vee f \vee \neg a \vee \neg d$	Watched by $\neg e, f$

Example of a clause containing $\neg e$ but not watched by it:
$f \vee ? \vee \neg e$

After deciding on e

(current assignement is a, b, c, d)

$\neg e \vee a \vee f \vee g$	Watched by $\neg e, a$
$b \vee f \vee \neg e$	Watched by $\neg e, f$
$f \vee \neg e \vee \neg a \vee \neg g$	Watched by $\neg e, f$
$\neg e \vee f \vee \neg a \vee \neg d$	Watched by $\neg e, f$

Deciding on e:

$\neg e \vee a \vee f \vee g$	Still watched by $\neg e, a$
$b \vee f \vee \neg e$	Watched by b, f
$f \vee \neg g \vee \neg a \vee \neg e$	Watched by $\neg g, f$
$\neg e \vee f \vee \neg a \vee \neg d$	Watched by $\neg e, f$... But UNARY !!

We have to cope with this lazy data structure

The state of the formula is unknown!

- How many reduced clauses? how many satisfied clauses?
- Some variables may be pure?

Only one guarantee: all unary and empty clauses are detected

Find a model? all the variables are assigned without any conflict

Heuristique : How to choose a variable to branch on? We are blind!

- We need to use the past, not the current state of the formula
- The heuristics will be heavily related to the clause learning mechanism

Ingredients of an efficient SAT solver

All ingredients are designed to cope with huge problems

Preprocessing
(and inprocessing)

Restarting

Branching

Conflict Analysis

Clause Database
Cleaning

CDCL principles at a glance

Decisions – Propagations

$$\phi_1 = x_1 \vee x_4$$
$$\phi_2 = x_1 \vee \overline{x_3} \vee \overline{x_8}$$
$$\phi_3 = x_1 \vee x_8 \vee x_{12}$$
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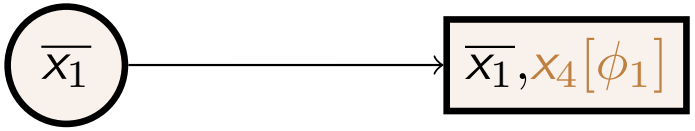
CDCL principles at a glance

Decisions – Propagations

DL 1



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CDCL principles at a glance

Decisions – Propagations

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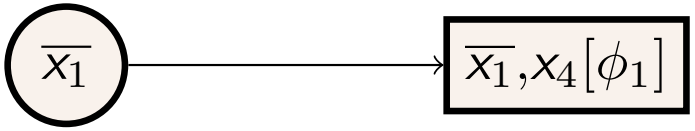

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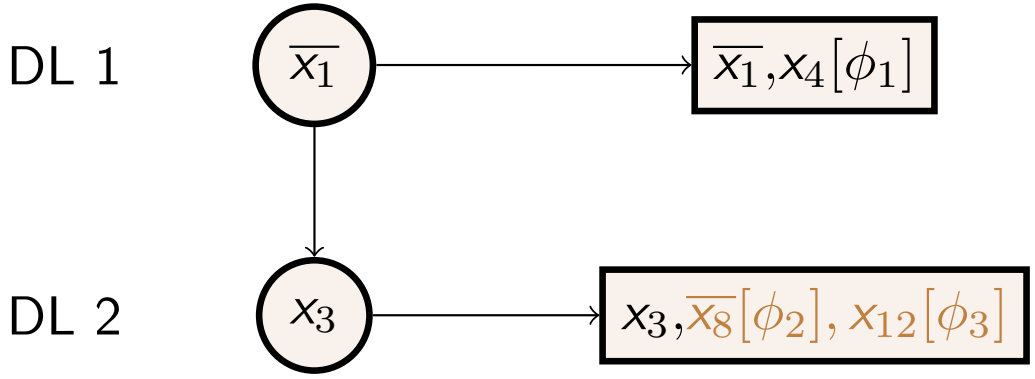

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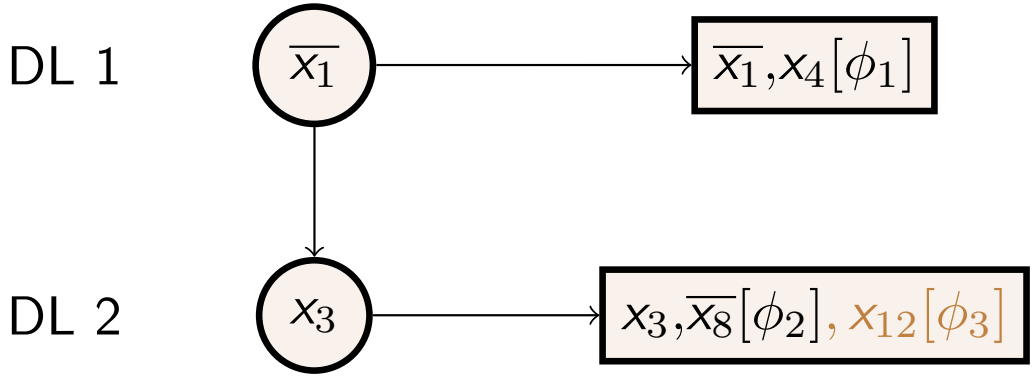

CDCL principles at a glance

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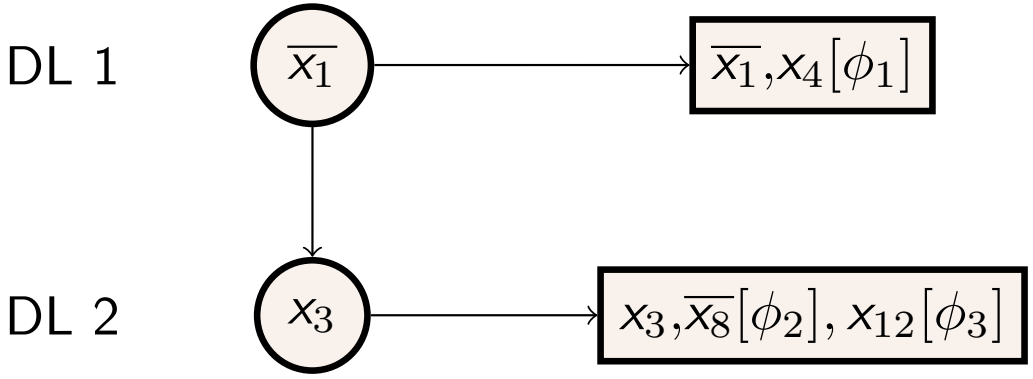

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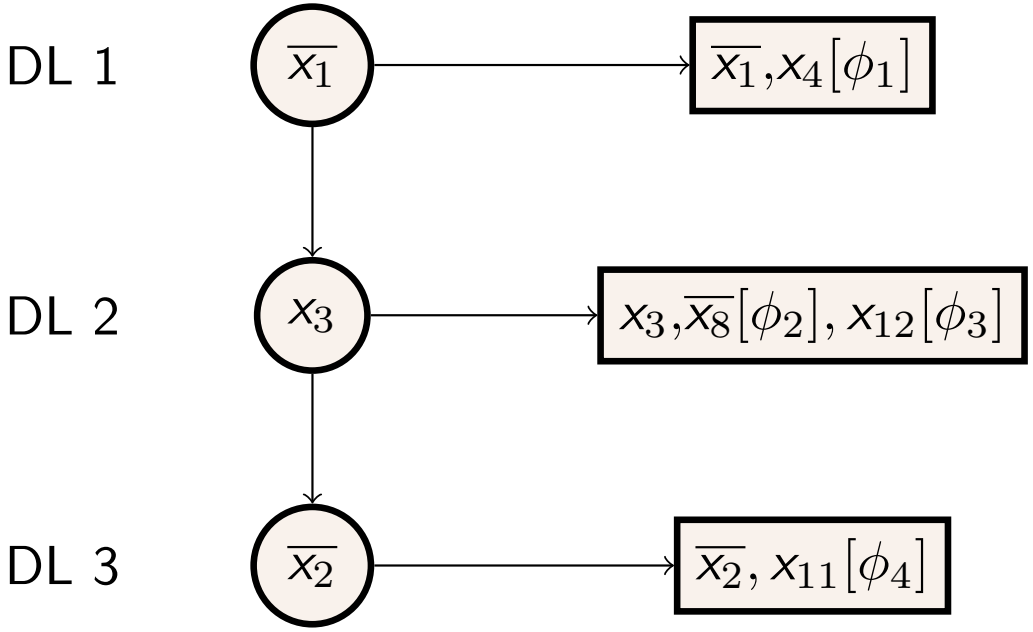

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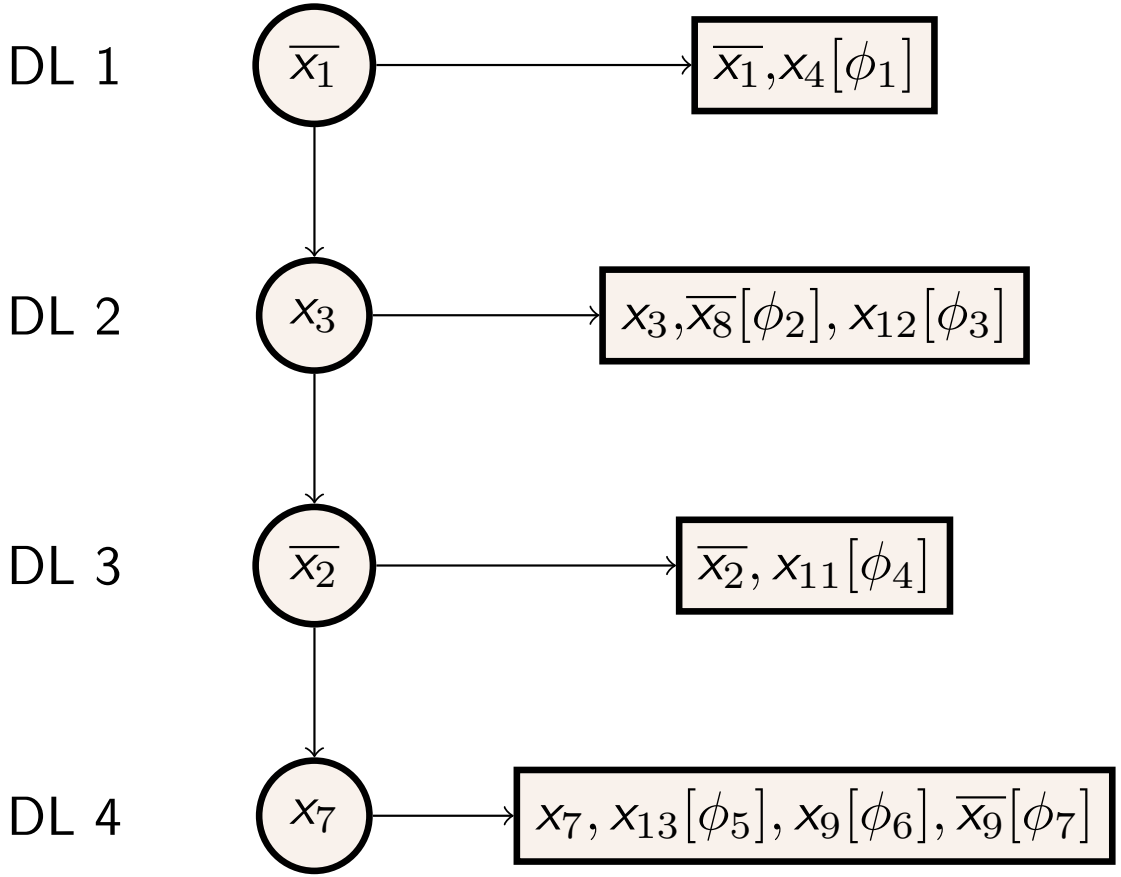
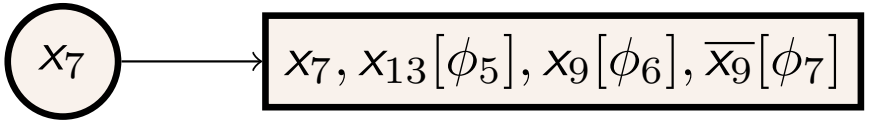


CDCL principles at a glance

Conflict Analysis

DL 4



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CDCL principles at a glance

Conflict Analysis

DL 4 $x_7 \longrightarrow \boxed{x_7, x_{13}[\phi_5], x_9[\phi_6], \overline{x_9}[\phi_7]}$

$$\beta_1 = res(x_9, \phi_7, \phi_6) = \overline{x_3} \vee x_8 \vee \overline{x_7} \vee \overline{x_{13}}$$

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CDCL principles at a glance

Conflict Analysis

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CDCL principles at a glance

Conflict Analysis

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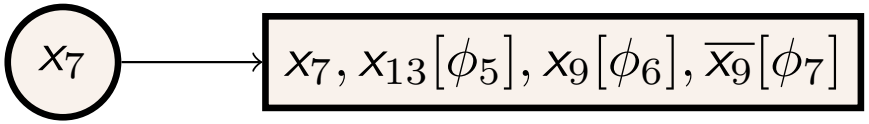

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$\beta = res(x_{13}, \beta_1, \phi_5) = \overline{x_3} \vee x_8 \vee \overline{x_7}$

- Stops as soon as the resolvant has a unique literal from the last decision level (FUIP).
- β is added to the clauses databases (ensure a systematic search)

CDCL principles at a glance

Non Chronological Backtrackings

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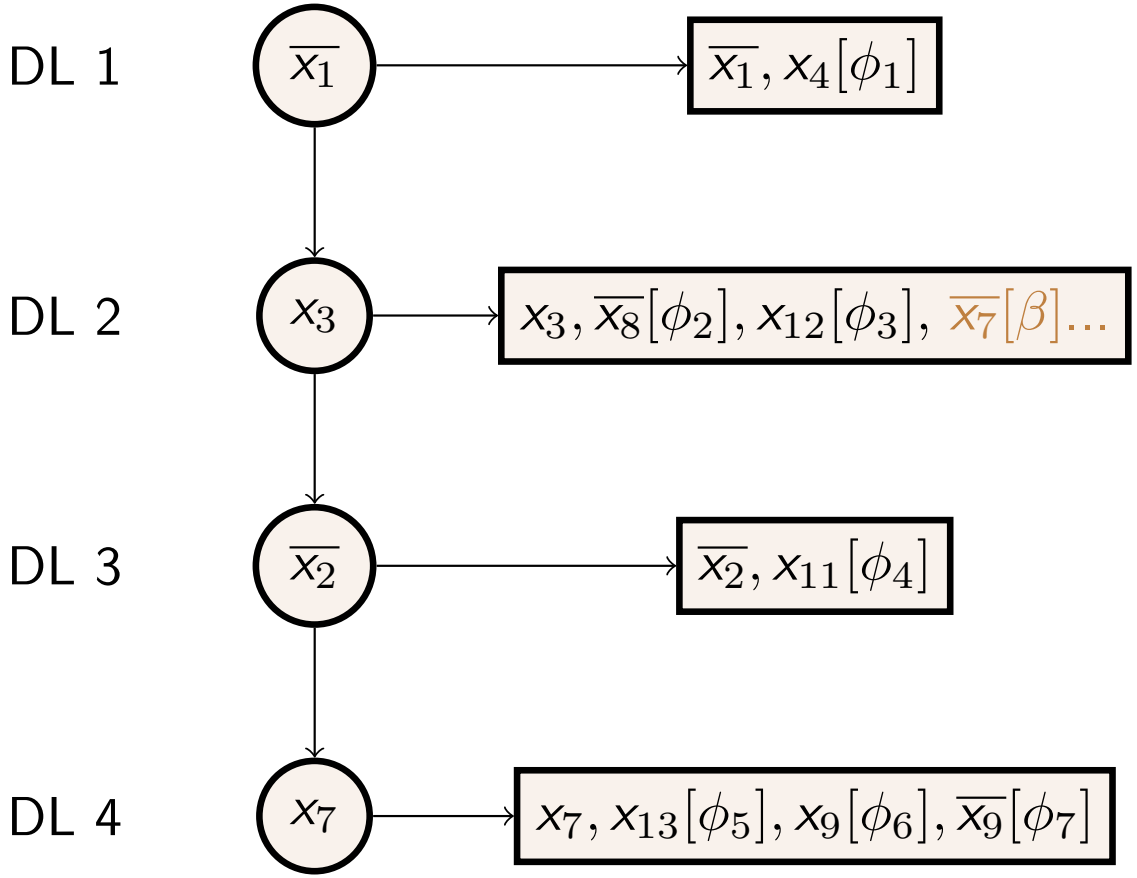

$$\beta = \overline{x_3} \vee x_8 \vee \overline{x_7}$$

CDCL principles at a glance

Non Chronological Backtrackings

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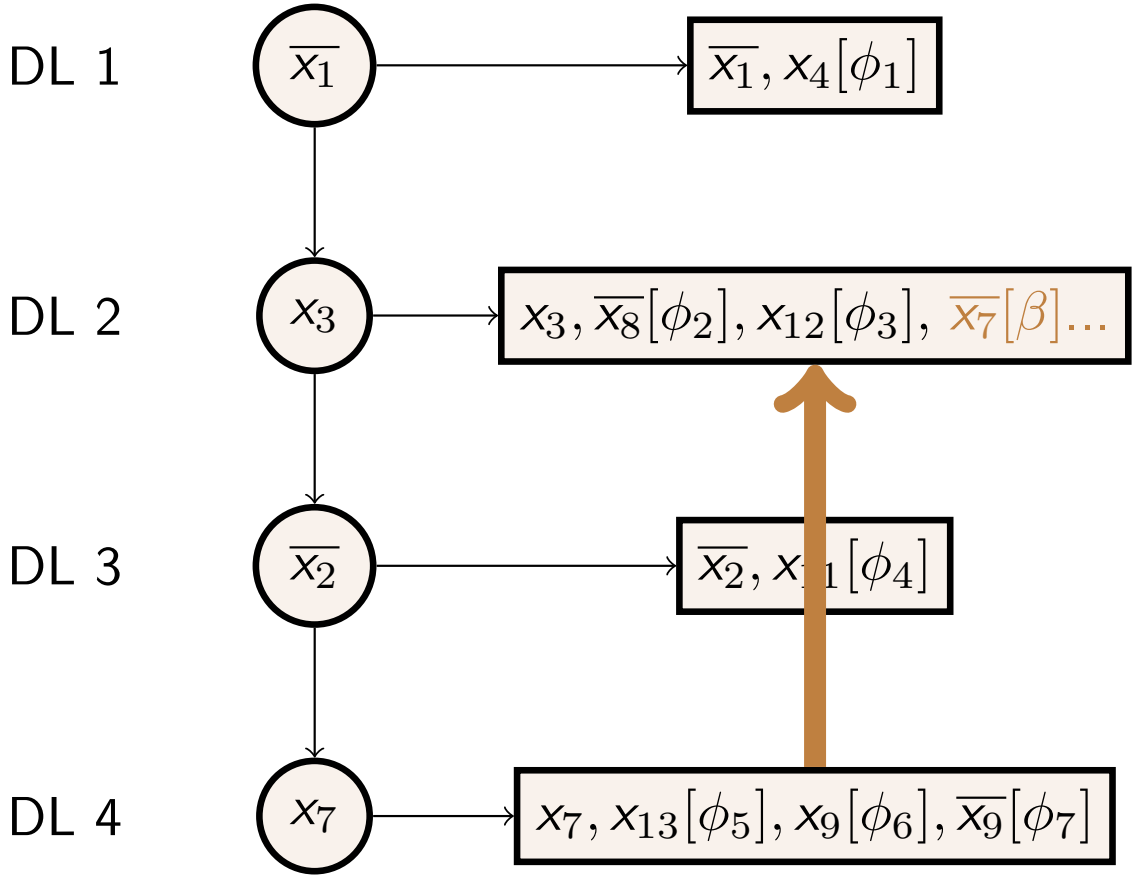

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CDCL principles at a glance

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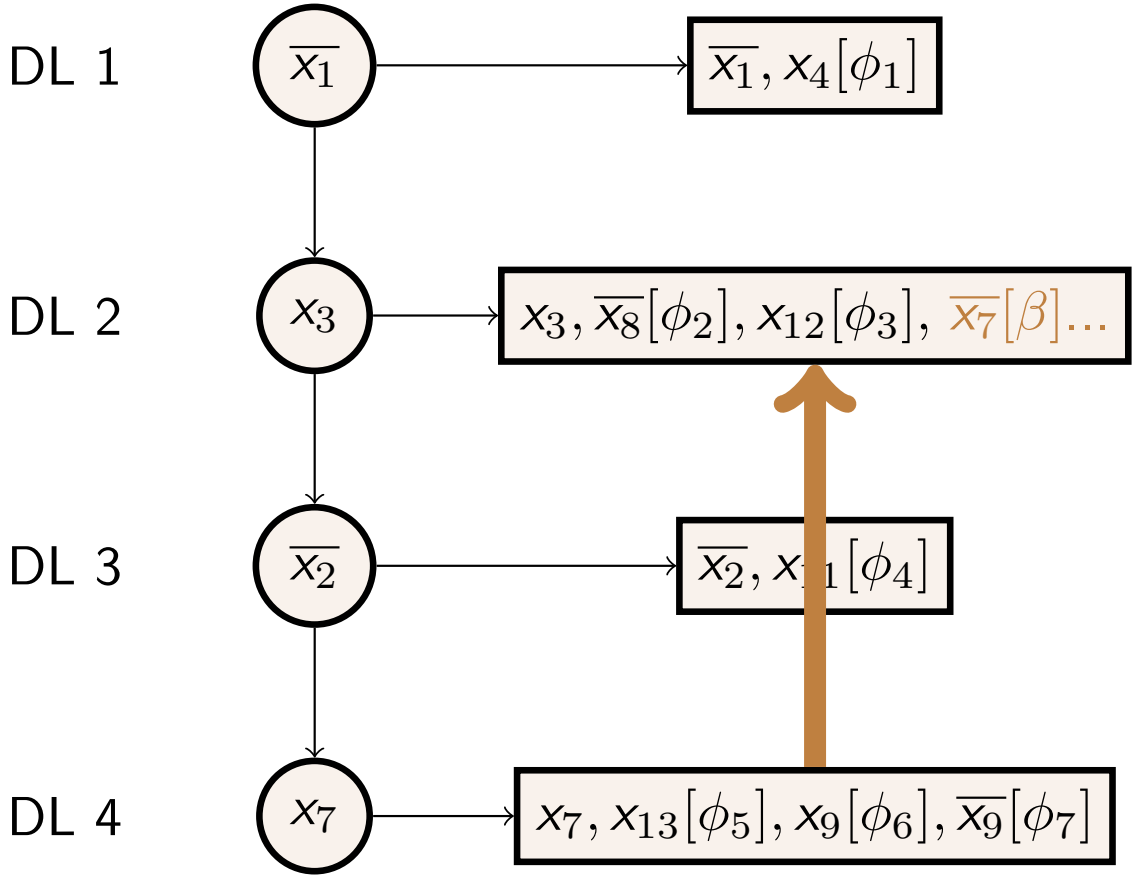

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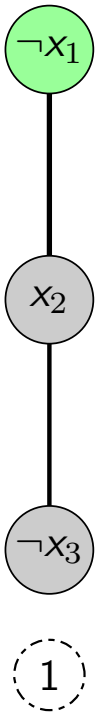
CDCL Principles, graphical point of view



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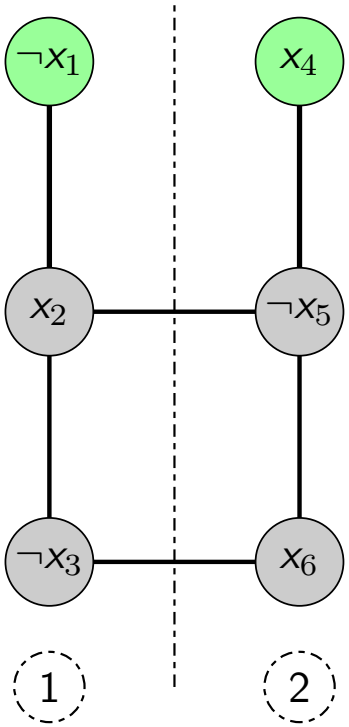
CDCL Principles, graphical point of view



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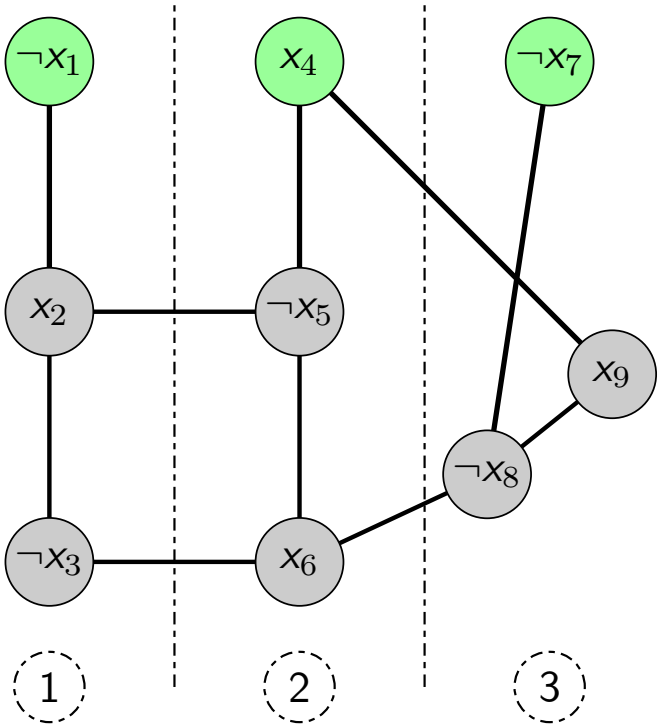

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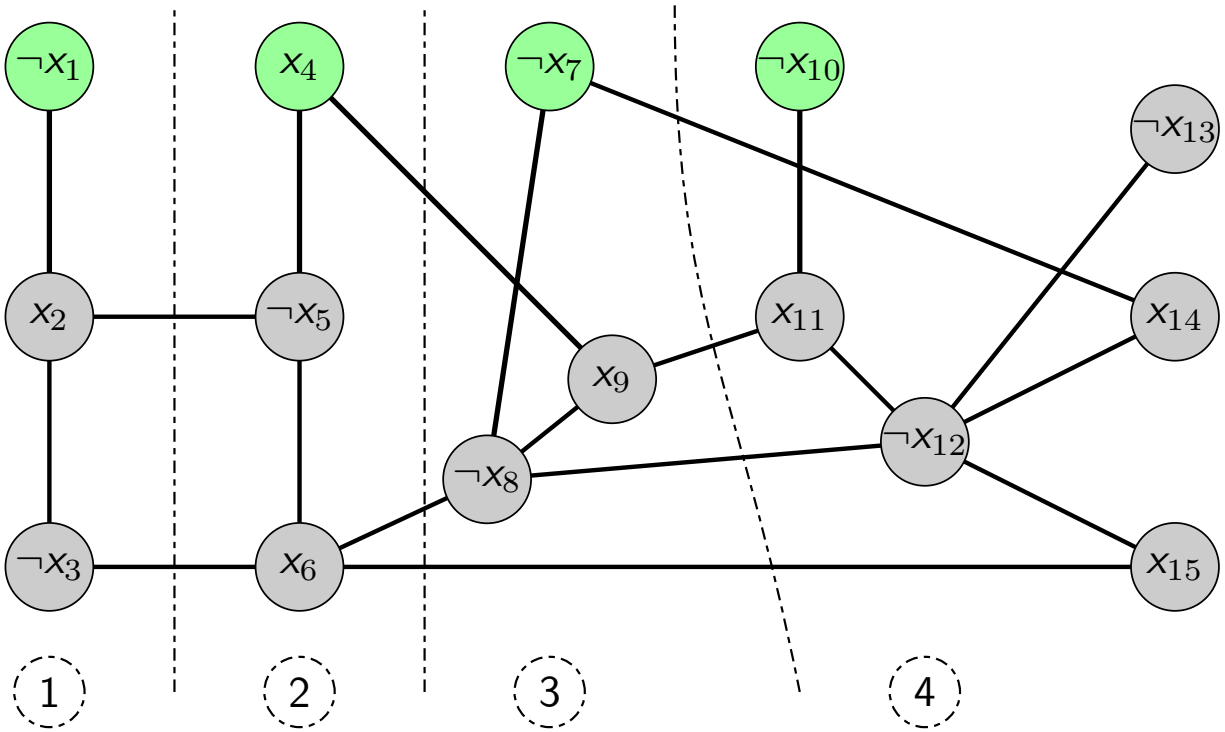

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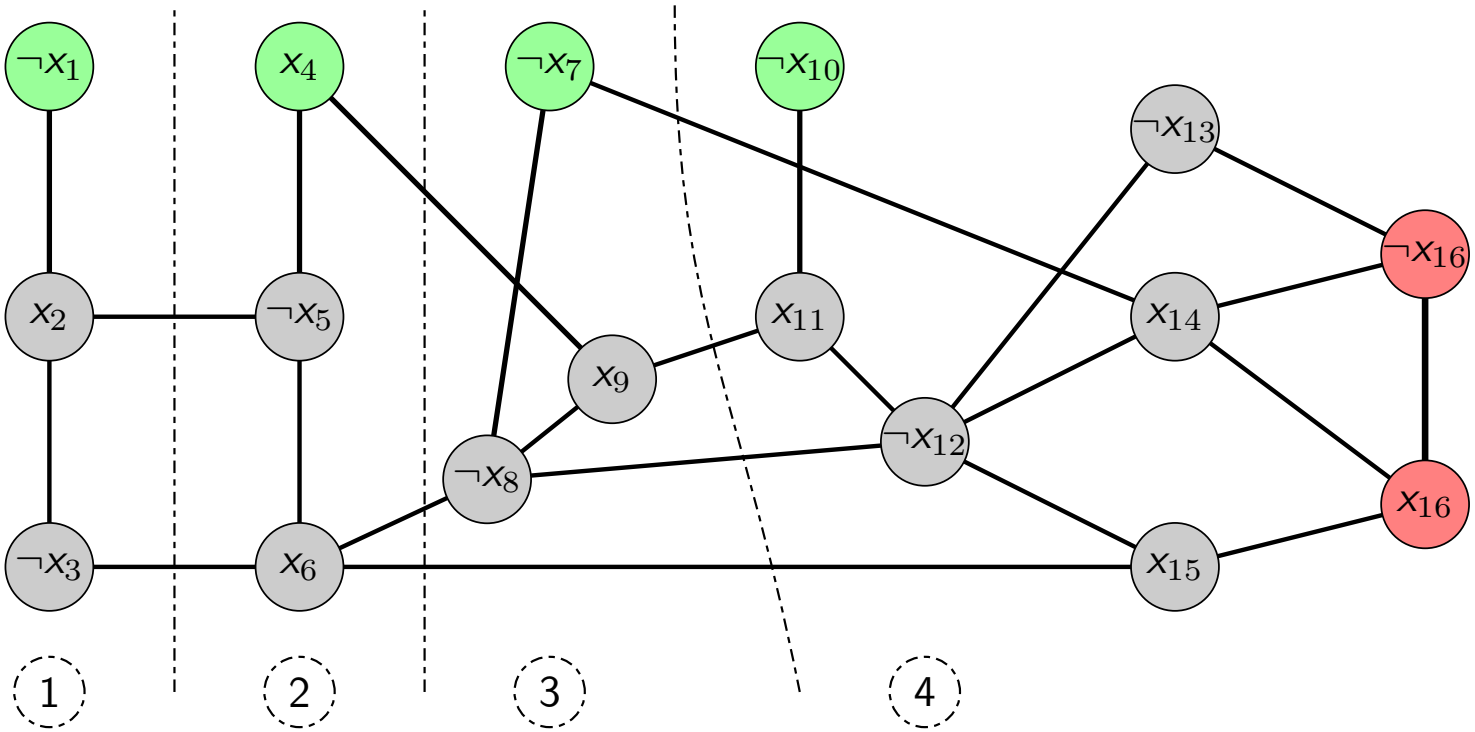

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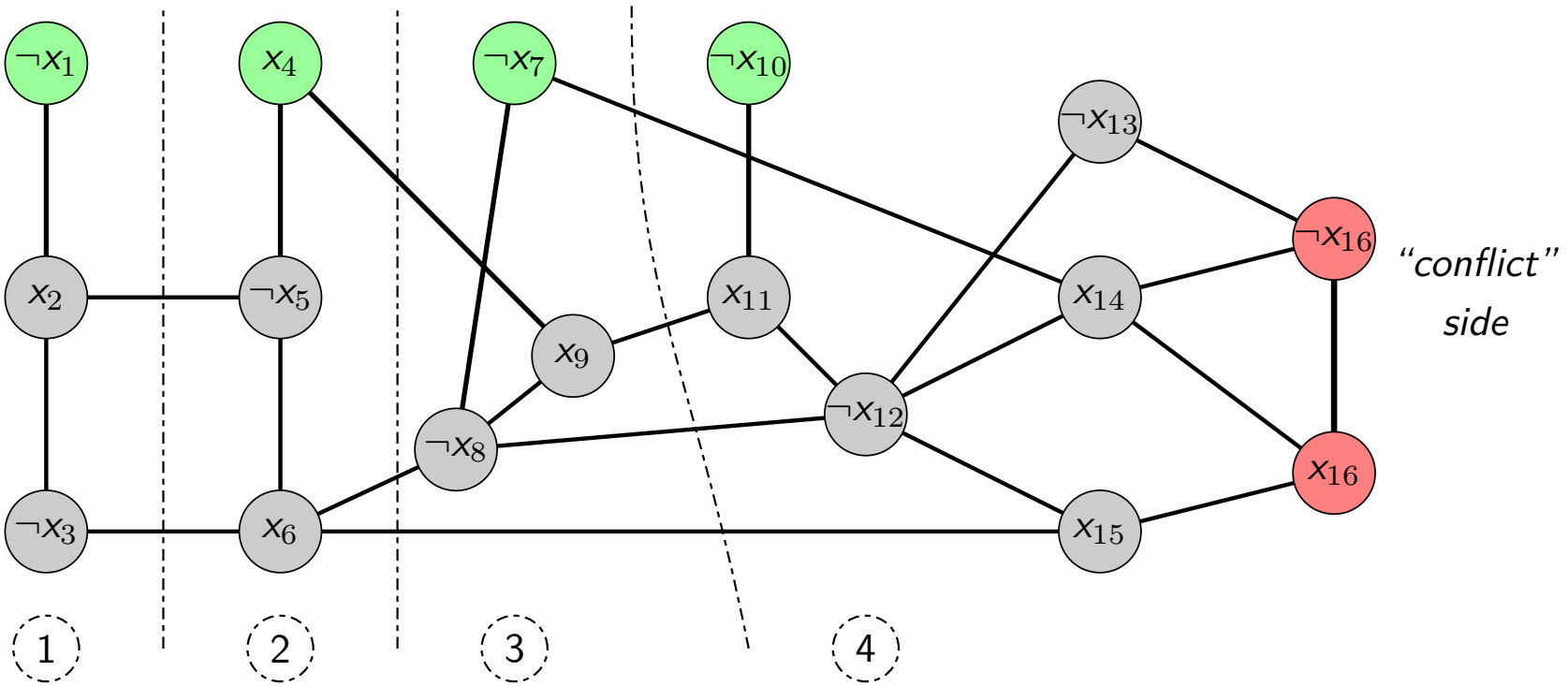

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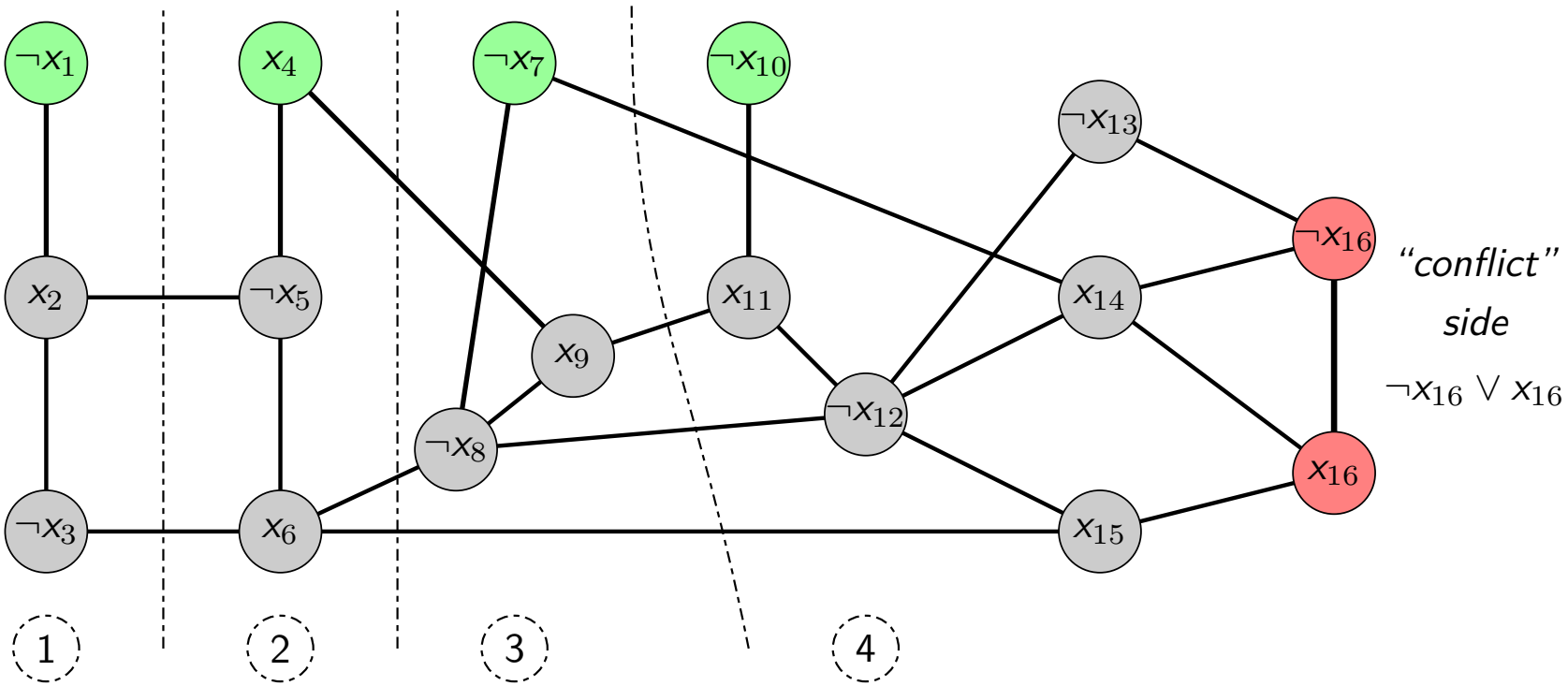

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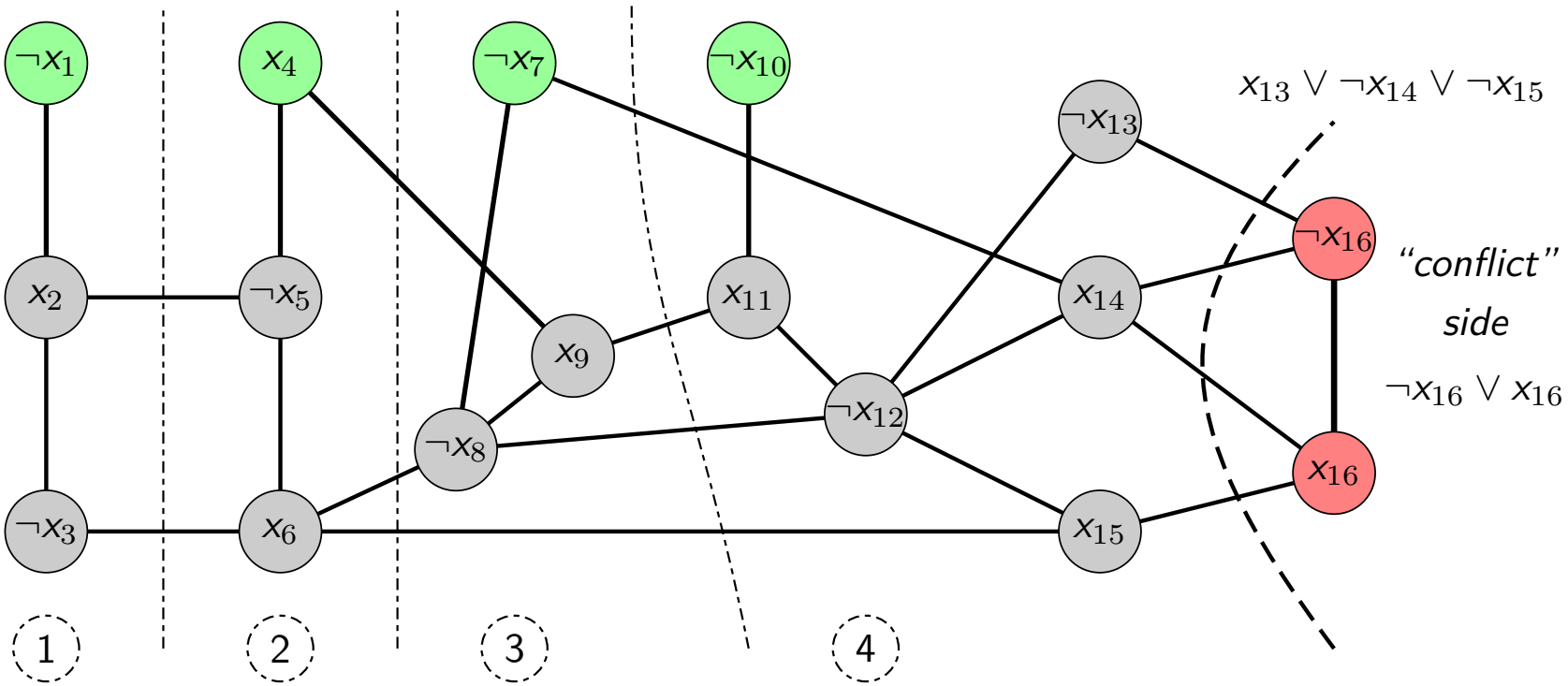
CDCL Principles, graphical point of view



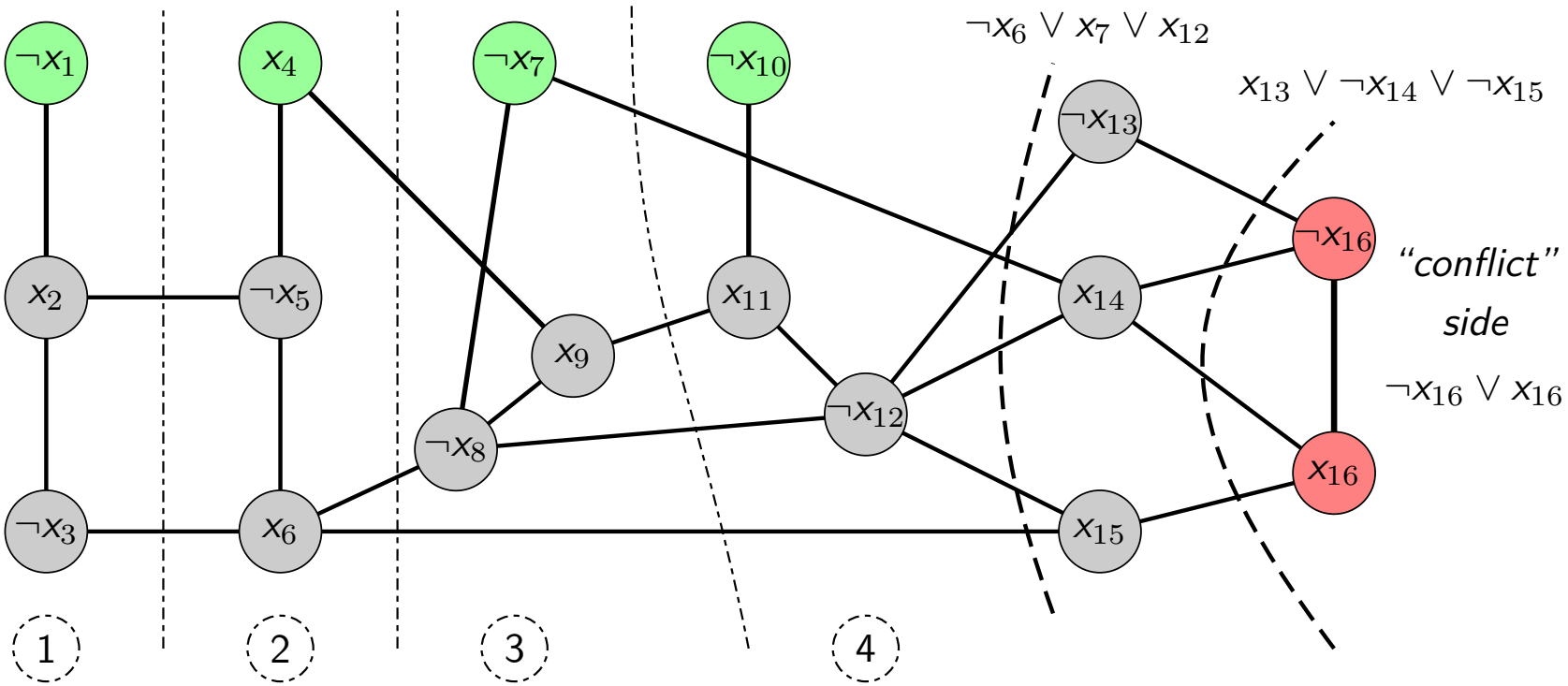
$x_1 \vee x_2$ $\quad$ $\neg x_2 \vee \neg x_4 \vee \neg x_5$ $\quad$ $x_7 \vee \neg x_6 \vee \neg x_8$ $\quad$ $x_{10} \vee \neg x_9 \vee x_{11}$ $\quad$ $\neg x_6 \vee x_{12} \vee x_{15}$

$\neg x_2 \vee \neg x_3$ $\quad$ $x_3 \vee x_5 \vee x_6$ $\quad$ $\neg x_4 \vee x_8 \vee x_9$ $\quad$ $\neg x_{11} \vee x_8 \vee \neg x_{12}$ $\quad$ $x_{13} \vee \neg x_{14} \vee \neg x_{16}$

$x_{12} \vee \neg x_{13}$ $\quad$ $\neg x_{15} \vee \neg x_{14} \vee x_{16}$

$x_7 \vee x_{12} \vee x_{14}$

CDCL Principles, graphical point of view



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$x_{10} \vee \neg x_9 \vee x_{11}$ $\quad$ $\neg x_{11} \vee x_8 \vee \neg x_{12}$ $\quad$ $x_{12} \vee \neg x_{13}$

$\neg x_6 \vee x_{12} \vee x_{15}$ $\quad$ $x_{13} \vee \neg x_{14} \vee \neg x_{16}$ $\quad$ $\neg x_{15} \vee \neg x_{14} \vee x_{16}$ $\quad$ $x_7 \vee x_{12} \vee x_{14}$

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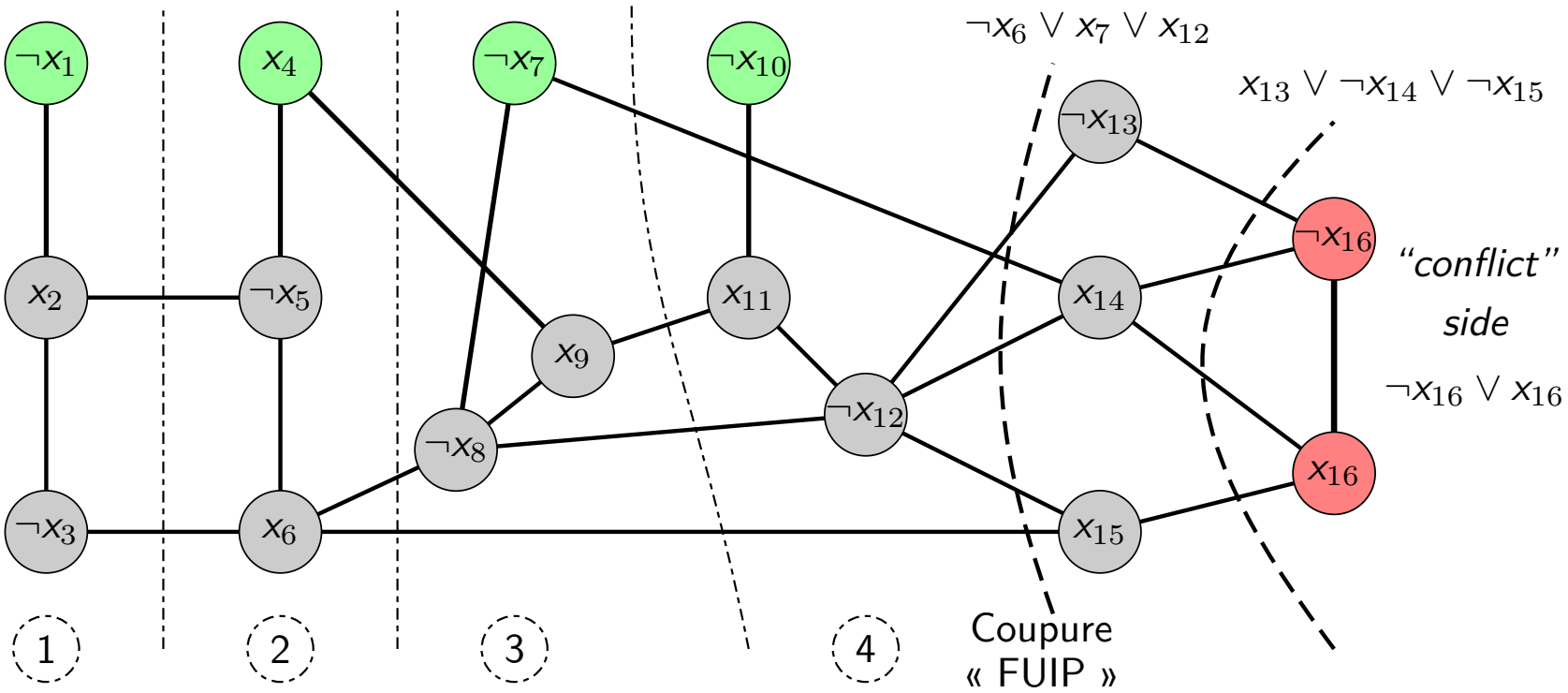

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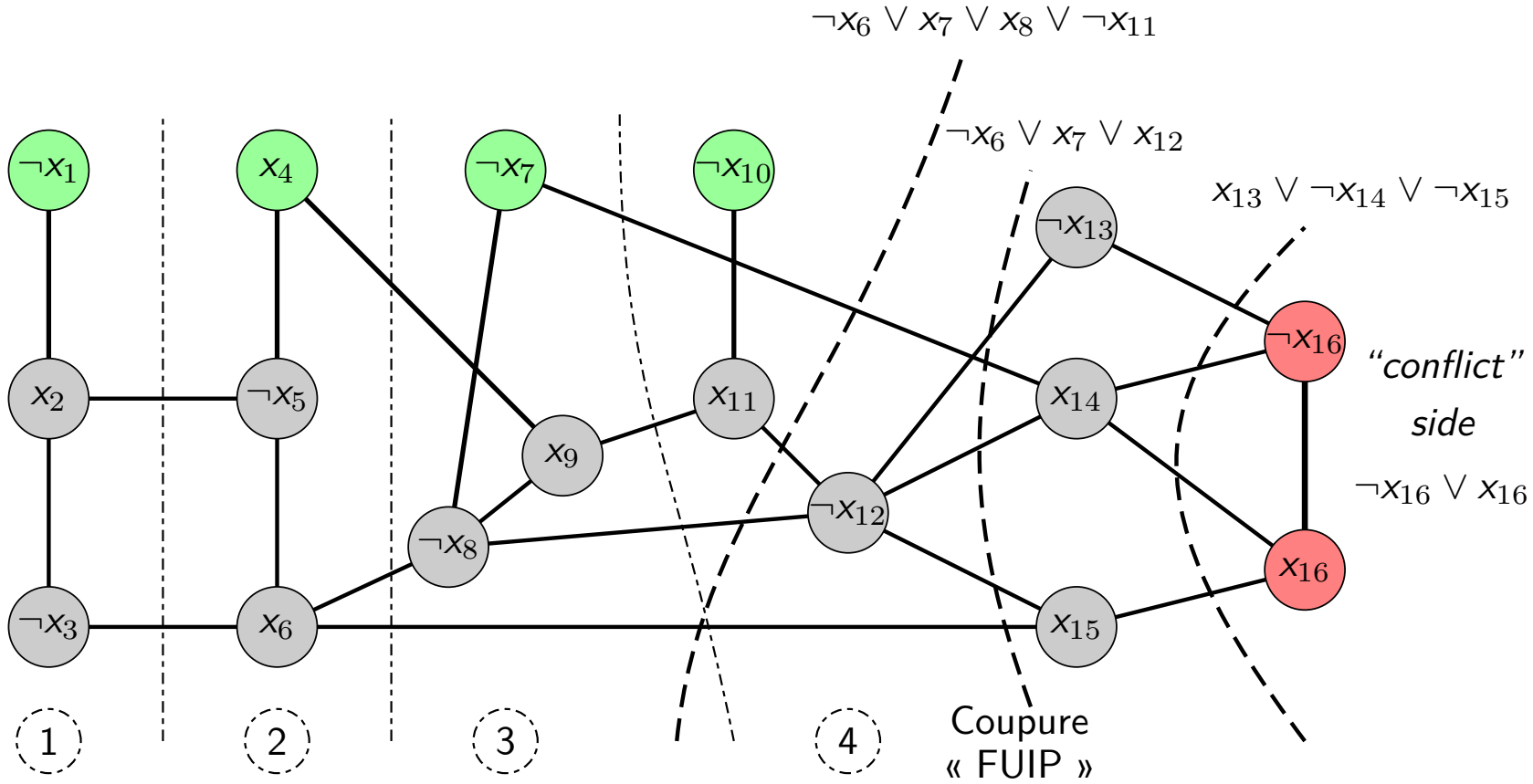

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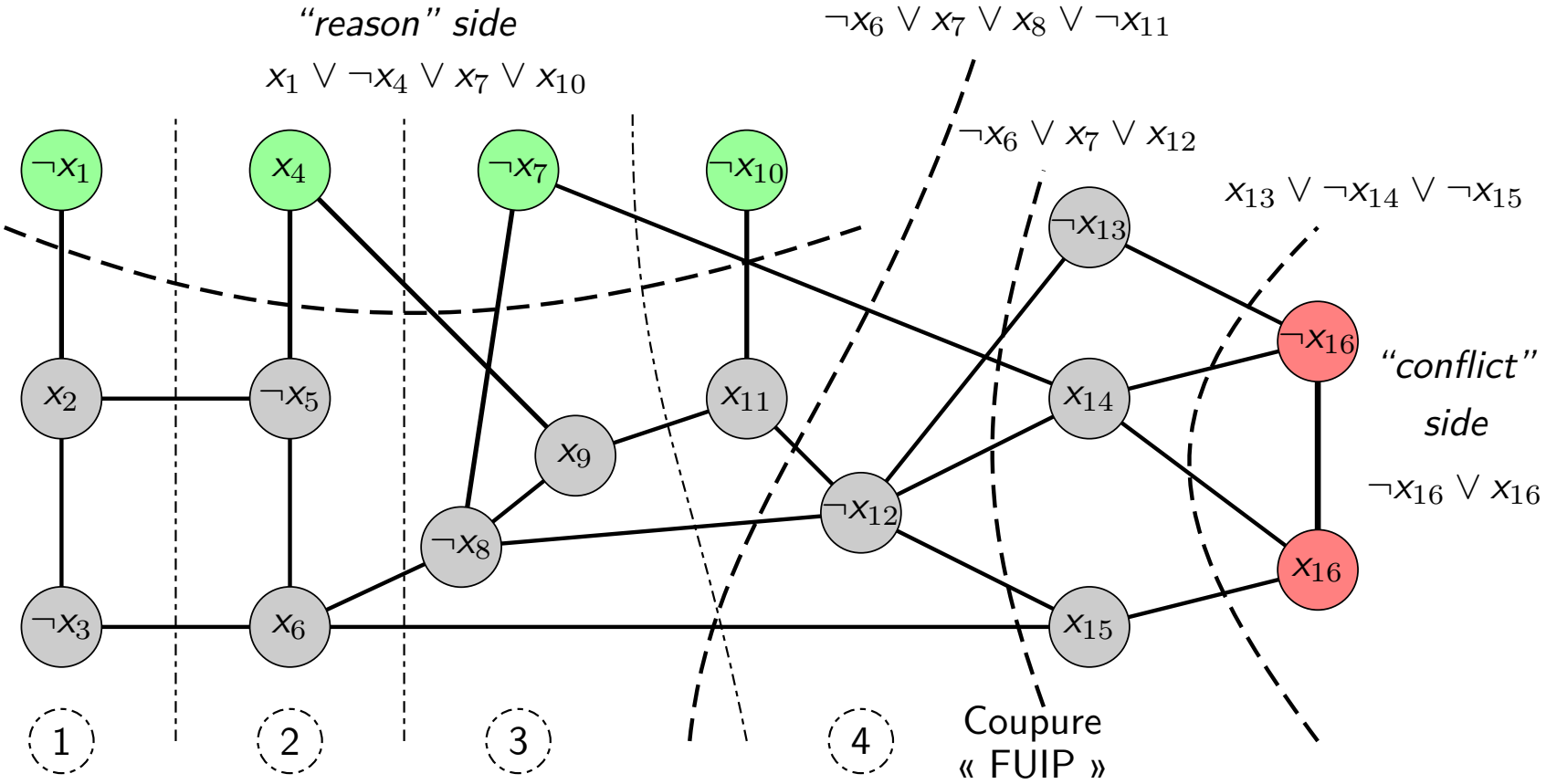

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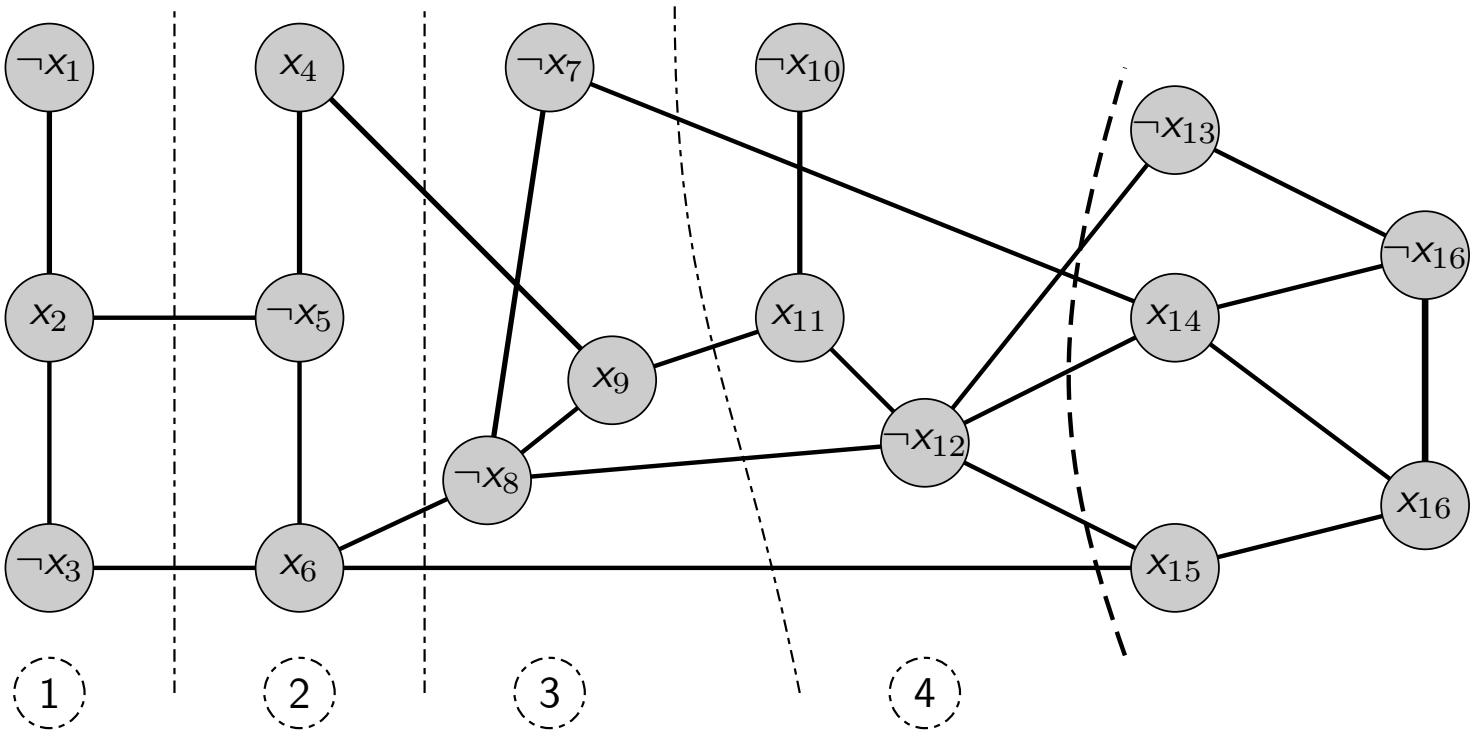

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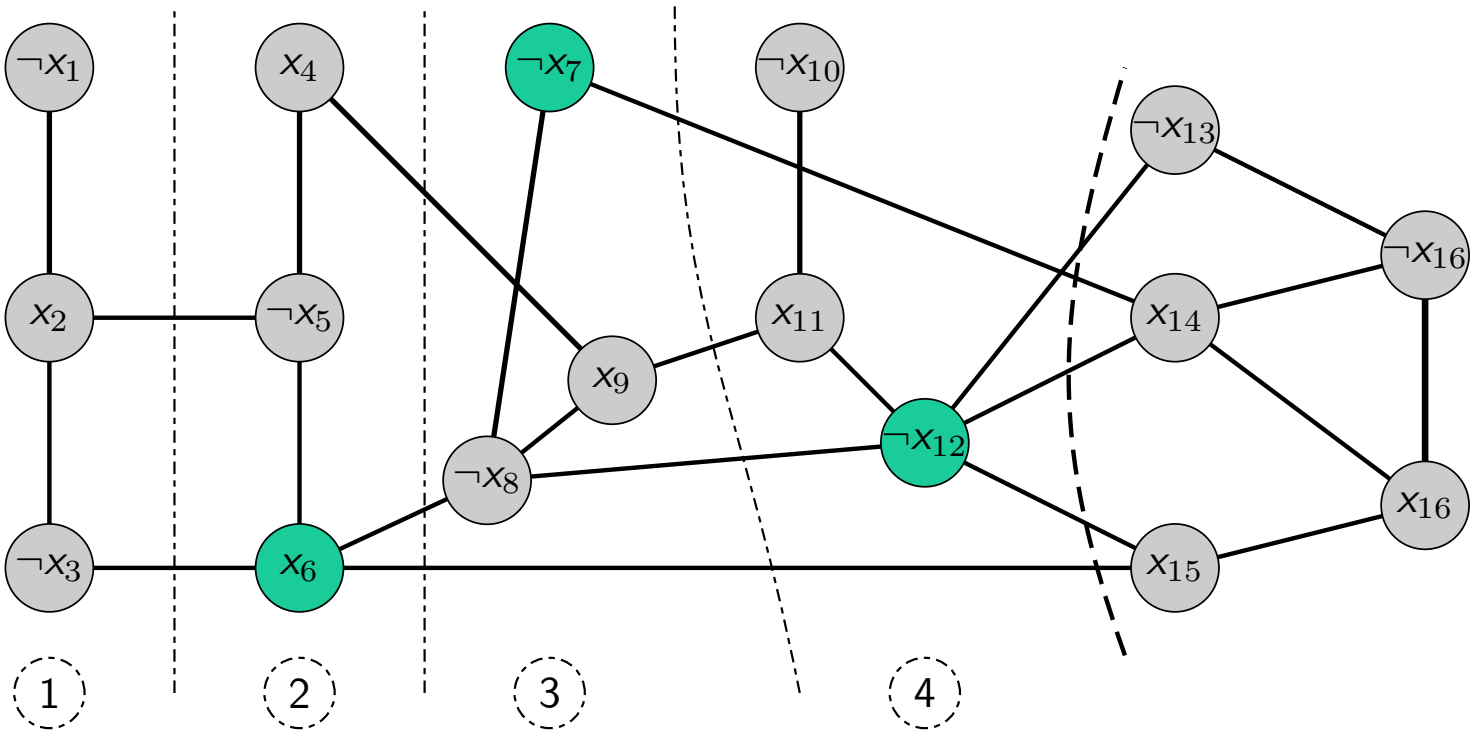

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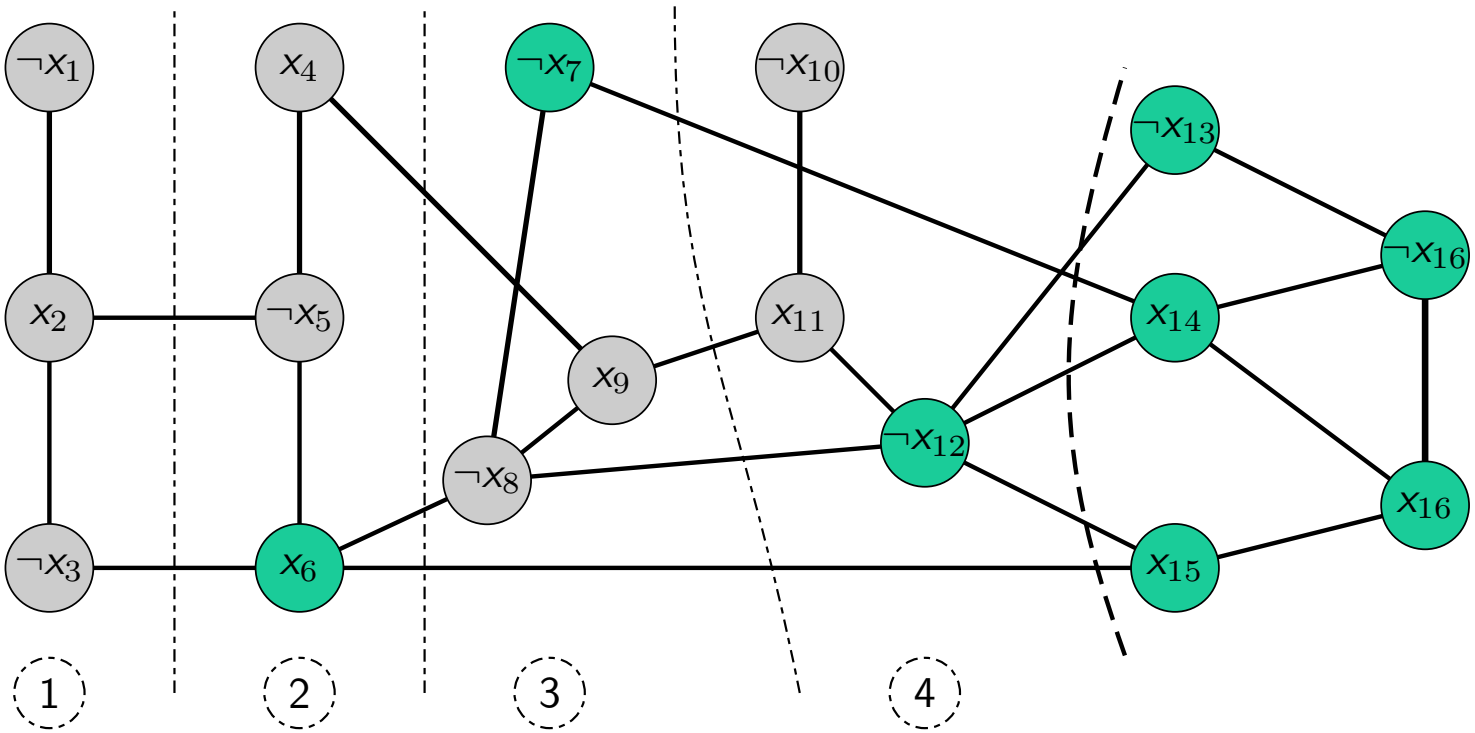

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Var	Lev	Val	Reason
x_1	1	$\bot$	$\emptyset$
x_2	1	$\top$	$x_1 \vee x_2$
x_3	1	$\bot$	$\neg x_2 \vee \neg x_3$
x_4	2	$\top$	$\emptyset$
x_5	2	$\bot$	$\neg x_2 \vee \neg x_4 \vee \neg x_5$
x_6	2	$\top$	$x_3 \vee x_5 \vee x_6$
x_7	3	$\bot$	$\emptyset$
x_8	3	$\bot$	$x_7 \vee \neg x_6 \vee \neg x_8$
x_9	3	$\top$	$\neg x_4 \vee x_8 \vee x_9$
x_{10}	4	$\bot$	$\emptyset$
x_{11}	4	$\top$	$x_{10} \vee \neg x_9 \vee x_{11}$
x_{12}	4	$\bot$	$\neg x_{11} \vee x_8 \vee \neg x_{12}$
x_{13}	4	$\bot$	$x_{12} \vee \neg \neg x_{13}$
x_{14}	4	$\top$	$x_7 \vee x_{12} \vee x_{14}$
x_{15}	4	$\top$	$\neg x_6 \vee x_{12} \vee x_{15}$
x_{16}	4	$\bot$	$x_{13} \vee \neg x_{14} \vee \neg \neg x_{16}$

Conflict when propagating

$x_{16} \vee \neg x_{15} \vee \neg x_{14}$

Let's do the conflict analysis.

$x_{16} \vee \neg x_{15} \vee \neg x_{14}$

$\neg x_{15} \vee \neg x_{14} \vee x_{13}$

$\neg x_{14} \vee x_{13} \vee x_{12} \vee \neg x_6$

$x_{13} \vee x_{12} \vee x_7 \vee \neg x_6$

$x_{12} \vee x_7 \vee \neg x_6$

Simply follows the order of the trail!

VSIDS : the killing heuristics

Variable State Independant Decaying Sum

Idea : The heuristics is state independant

Award variables (not literals) view

- in the most recent learnt clauses
- during the last conflict analysis

Well, yes...But where to branch at the beginning?

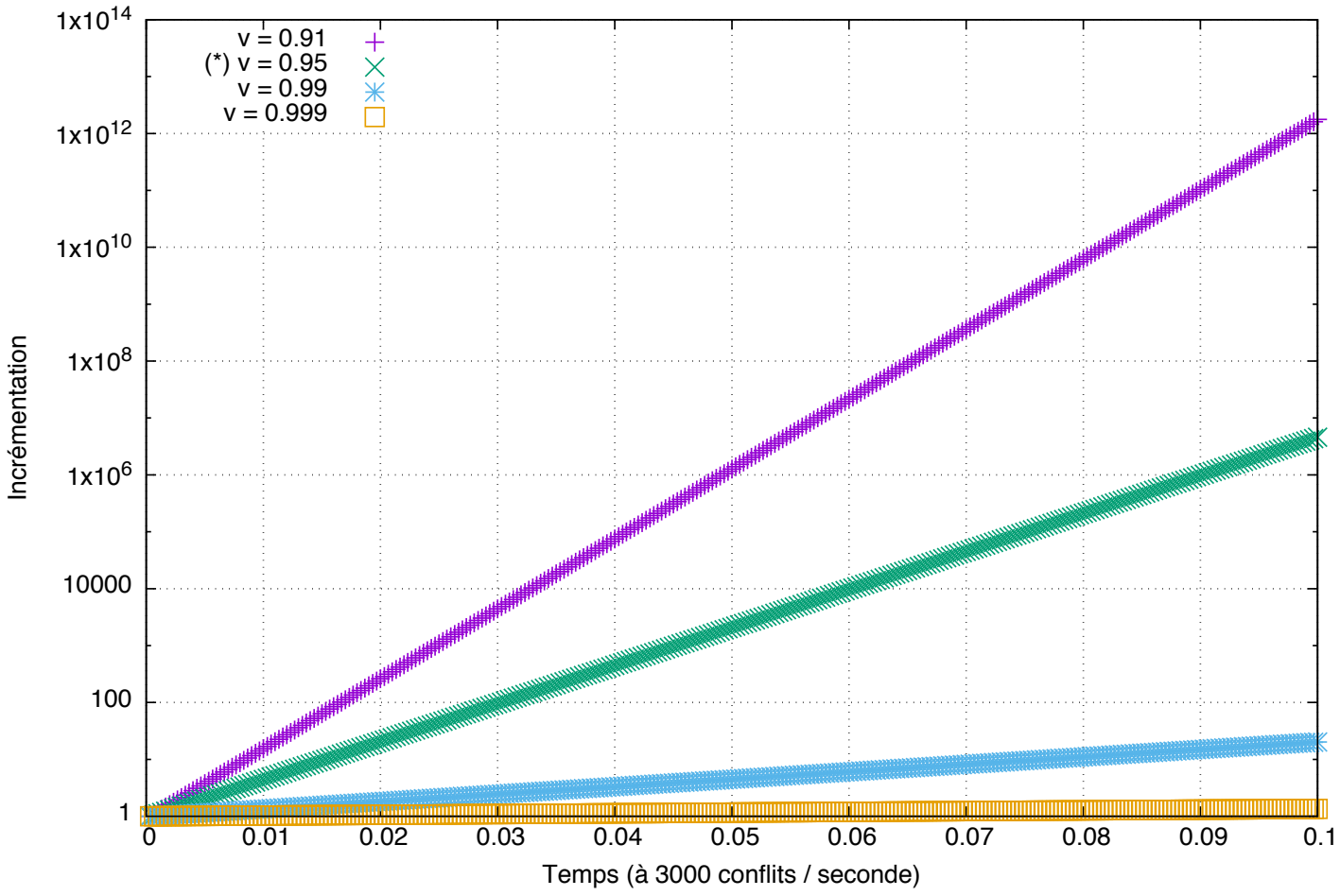
The heuristics is **highly dynamic (really, really)**

- in 1/30s (not in Python) a variable can fall from the top to the abysses

Side effect: the solver is very hard to understand/predict

VSIDS dynamicity

Exponential increasing of the increment



After each conflict, it is multiplied by $1/v$ ($v = 0.95$)

VSIDS and the heap

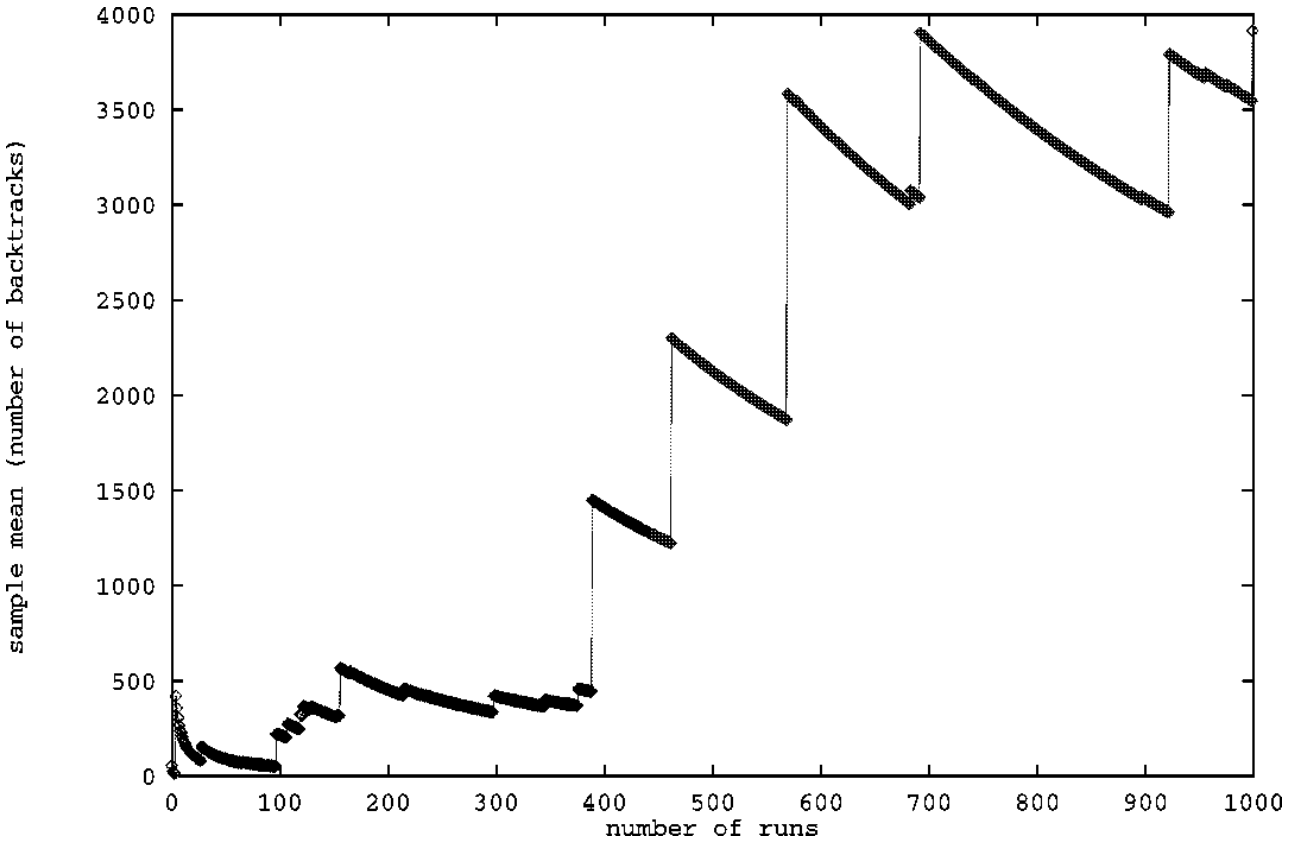
Some tricks:

- Uses a heap for sorting variables
- When a variable is chosen by the heuristics, remove it from the heap (free)
- When a variable is propagated, don't remove it from the heap (let's be lazy)
- Thus, picks variables from the heap until it is unassigned
- Push back variables to the heap when backtracking

Restarts, common idea

Restarts are commonly used in local search algorithms.
Idea: when the search failed, look at another part of the search space (looking for a solution).

Heavy Tailed Phenomenon Observed on Search Trees



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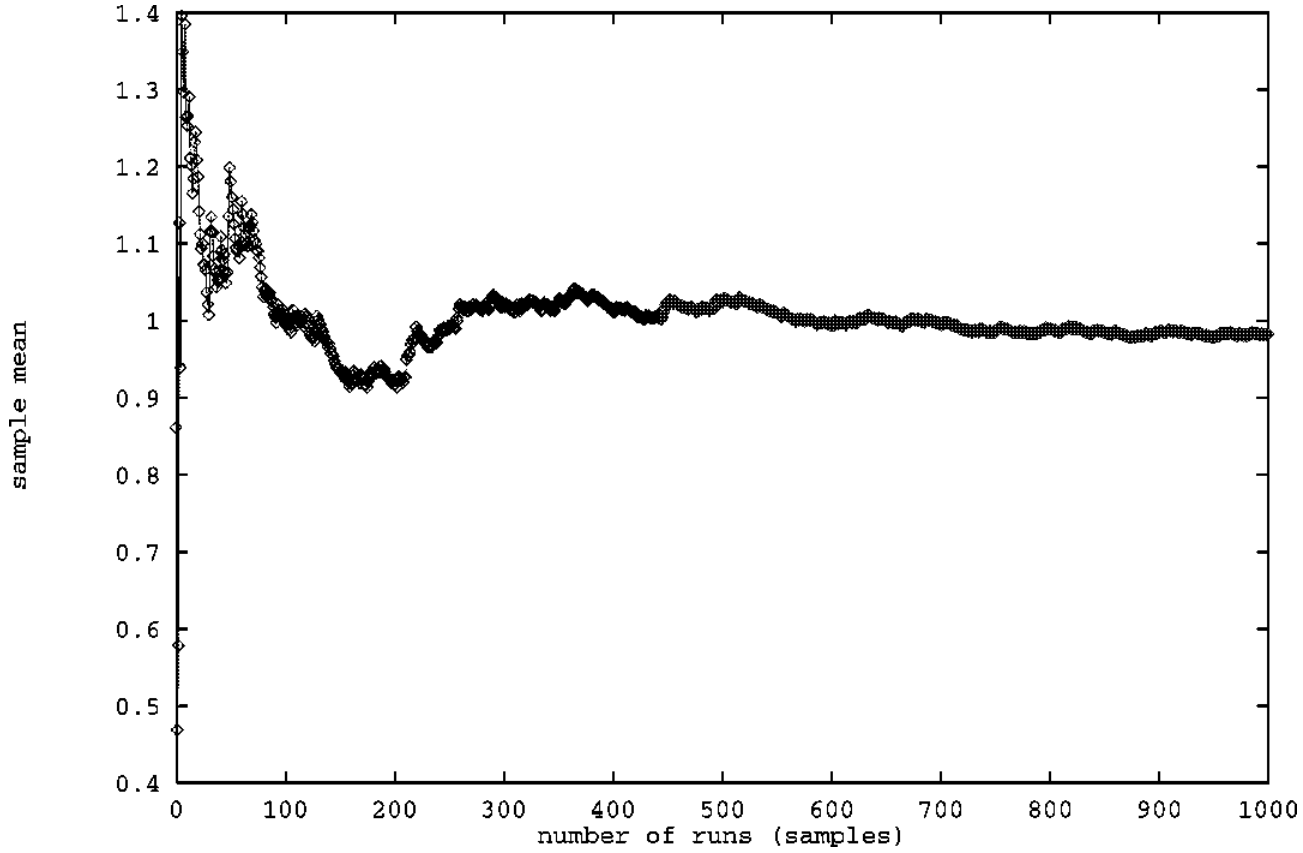

Restarts, common idea

Restarts are commonly used in local search algorithms.
Idea: when the search failed, look at another part of the search space (looking for a solution).

Restarts are introduced to fight this phenomenon

Note: Learning allows frequent restarts. The completeness is still guarantee.

Now: Ultra-Rapid Restarts

Last progress Luby-law restarts
(1,2,1,2,4,1,2,1,2,4,8,1,2,1,2,4,1,2,1,2,4,8,16,...)
Multiplied by a constant (32, 64, 128, 512).

Other strategies decision levels / LBD based / agility
Idea When the solver does not make any progress, restart

Important thing: restart a lot (optimal: Luby 6), use in conjonction with phase-saving

Restarts do not restart search Heuristics are maintained during a restarts. The solver directly goes to the same search space, but with a distinct path.

Now: Ultra-Rapid Restarts

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Many other ingredients

Conflict Clause Minimization

Phase Saving

Blocking Literals

Pre-processing

In-Processing

...

Clause Database Management

In the early 2000, Chaff was dying because of memory
(very quickly)
(some) learnt clauses had to be removed

- Regularly remove unseen clauses during last conflict analysis
- Extends the notion of VSIDS to clauses

Keeping as many clauses as possible was believed to be crucial

- Essential for completeness!
- Especially if we restart a lot!

Glucose: the last SAT surprize

We can **remove 95% of learnt clauses** and get better!

Instead of removing clauses when needed, we eagerly remove them very often

Remove half of the clause at each point of a dynamic strategy

Clause activity is not relevant. **Use the notion of LBD instead**

Literal Block Distance (LBD) – initial idea (2009)

- One decision often creates a lot of propagated literals ("blocks")
  ➔ Those variables will probably be propagated together again and again
- Reducing decisions? Adds dependencies between independent blocks
- How? Add the strongest possible constraints between them

LBD of a learnt clause: number of prop. blocks of literals

- Small LBD scores are better
- The importance of "Glue Clauses" (LBD=2)
  - Only one literal from the last decision level (the assertive one)
  - This literal will be **glued** to the other block
  - Kept forever in glucose

The restart policy is based on LBD too

What about parallel CDCL?

Bad idea: Try to parallelize the BCP engine

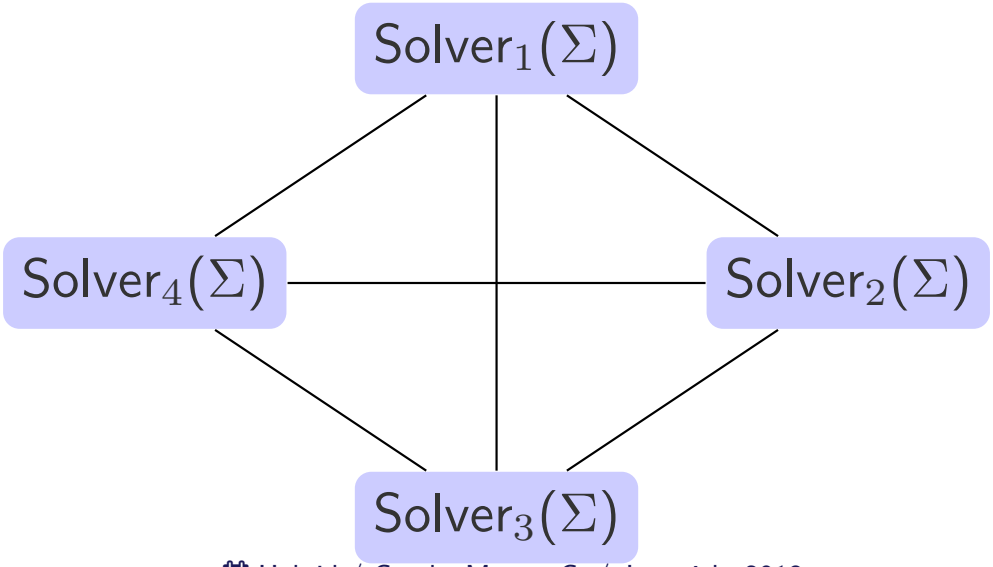
A very difficult task: each engine is eagerly using the memory

Even sharing a clause is not a good idea
The 2-Watched literal scheme is a nightmare for sharing the clause
in a single core engine the first 2 literals are a shared information between the 2 watched list.

So, the state of the art is to **duplicate** clauses

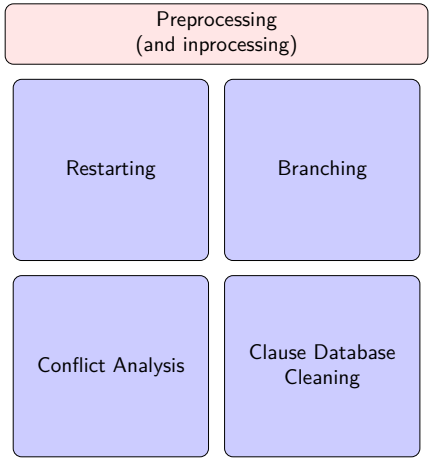
Many Cores

- Multi-core architecture, cloud ⇒ design of parallel solvers
- Different strategies
  - Divide and conquer
    ➔ Explicitly splits search w.r.t. assignments
  - Portfolio algorithms
    - Each thread: its own solver and the whole formula
      ➔ Rely on orthogonal searches
  - Parallel solver
    - communication: learnt clauses
      ➔ All threads working on the same proof



Many Cores, Many more problems

Sharing clauses in parallel has many drawbacks

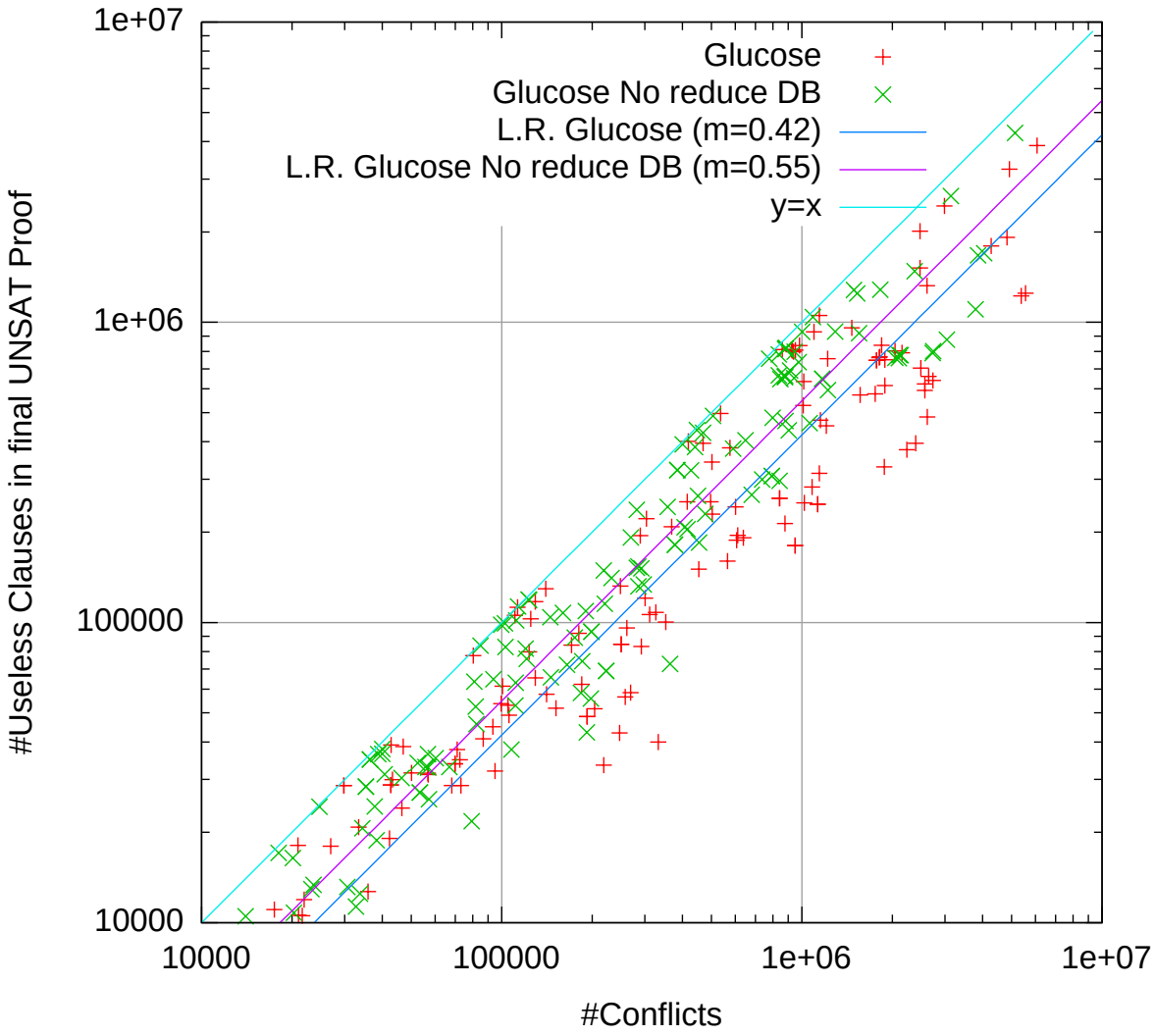


- Imported clauses can be bad (noise, wrong way, ...)
- Imported clauses can be subsumed / useless
- Imported clauses can dominate learnt clauses
- Each thread has to manage many more clauses
- Many side effects on all core components

Currently: Clauses are sent as soon as they are learnt, plus:

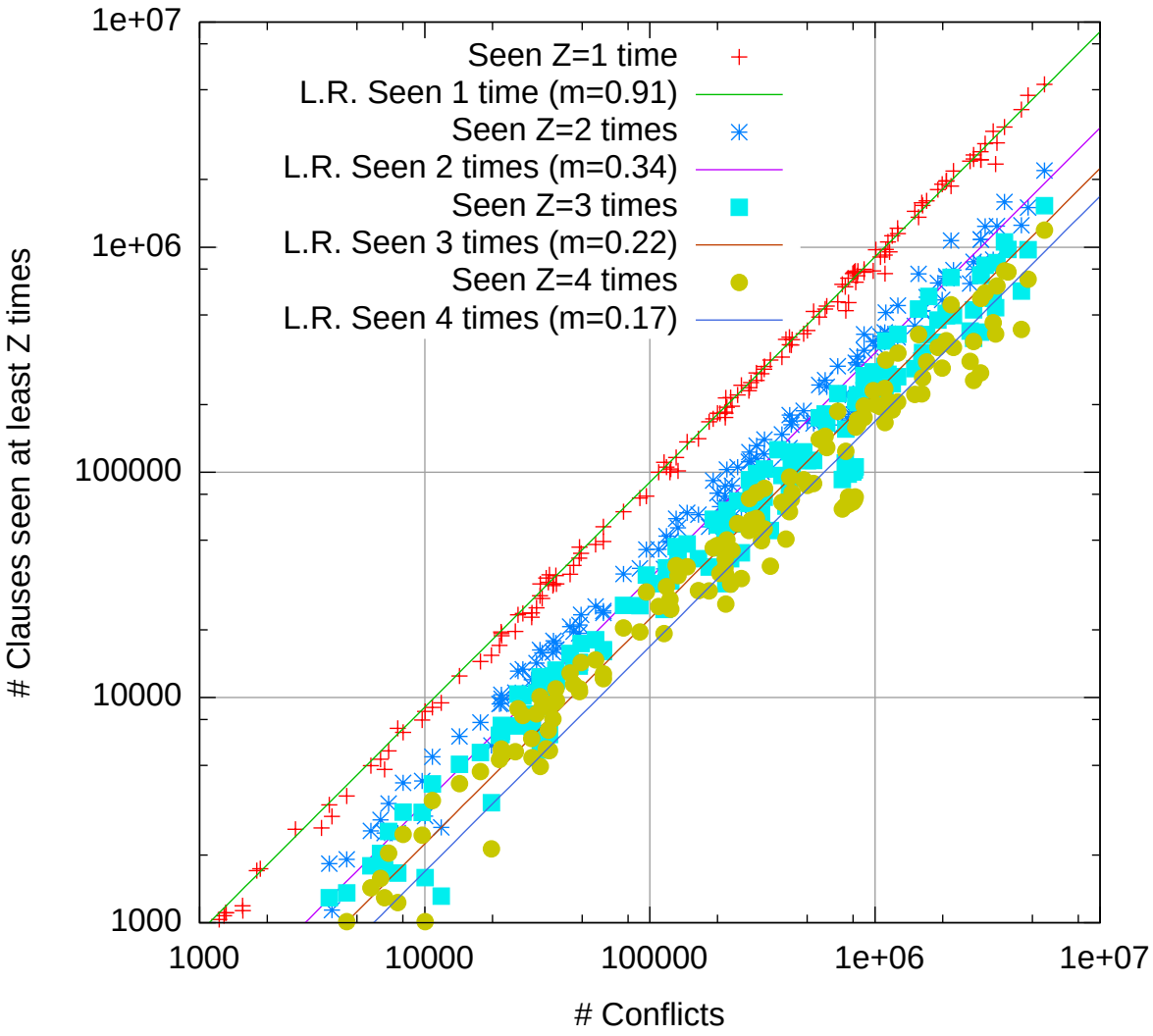
- MANYSAT 1.0: size ≤ 8
- MANYSAT 1.1: dynamically adjust the threshold
- PLINGELING: size ≤ 30 and LBD ≤ 8
- PENELOPE: LBD ≤ 8 (PSM allows more clauses exchanges)

Many *useless clauses* even in single engine solvers



- x-axis : Number of conflicts
- y-axis : Useless clauses in final proof (UNSAT formulas)

How many times clauses are seen in conflicts



- x-axis: Number of conflicts
- y-axis: Number of clauses seen at least Z times

Today's Itinerary

Do we really understand what we have built?

We know how to build an efficient (single engine) SAT solver, but:

- CDCL is not DPLL
  - ➔ *because of ultra-rapid restarts and agressive clause DB cleaning*
- Learning can be bad
  - ➔ *we'll see that keeping all clauses is not a winning strategy*
- Restarting is not restarting
  - ➔ *directly go to the same search space, by an another path*
- Luby-based restarts are dangerous
  - ➔ *rare but very large windows are following a fixed restart strategy*
- What is the phase?
  - ➔ *good to reach a solution or a contradiction?*
- "good" variables: top or bottom of the tree?
  - ➔ *splitting on top, resolving on bottom variables*
- glucose is not complete! ➔ *Too many restarts and forgetting*

CDCL solvers are complex systems

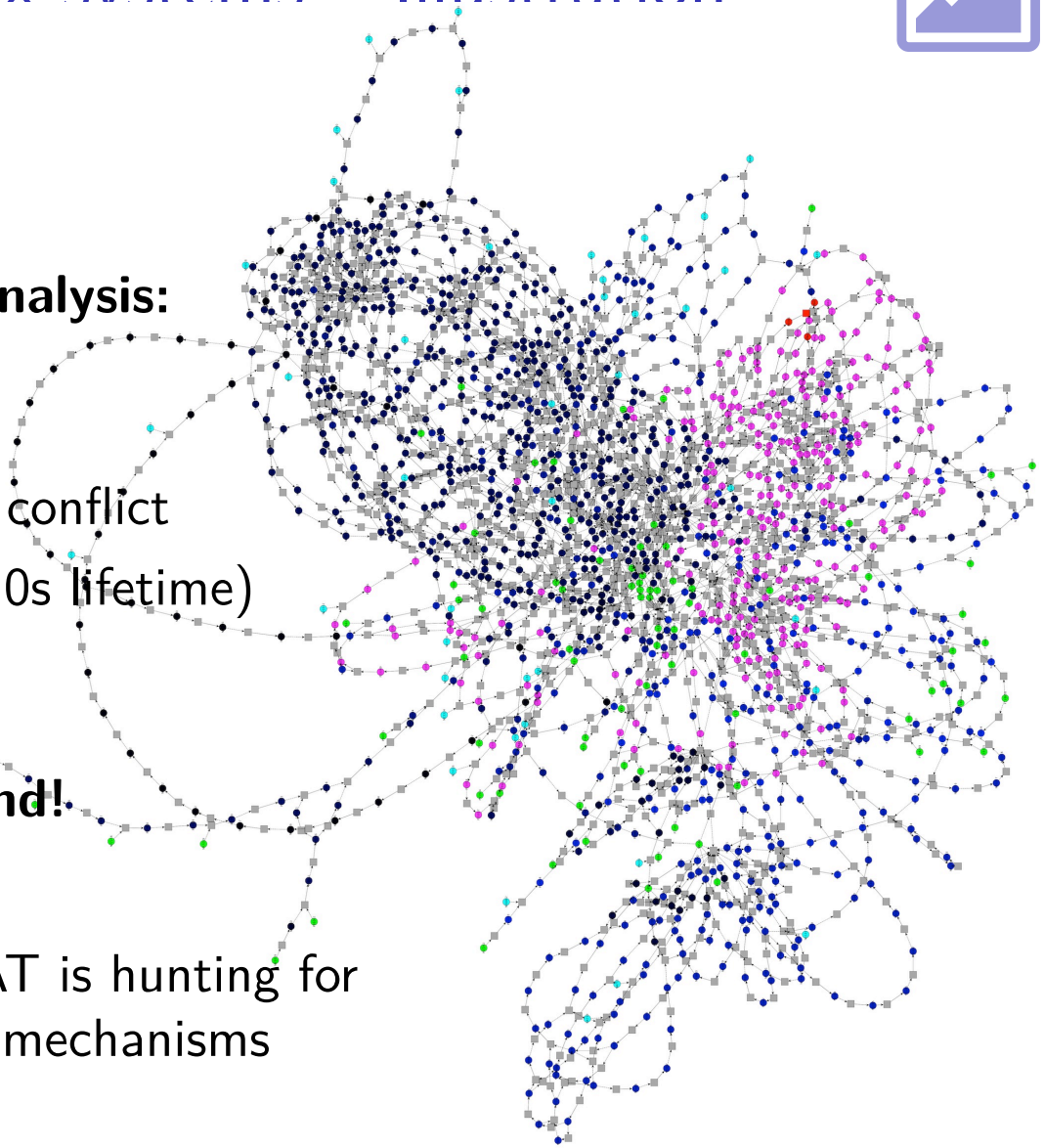
We have a lot of open problems around these questions:
“ Understand what we have implemented ”

It's ok if we don't fully “understand” our code

- Very fast and unpredictable
- Work well on real-world instances, but how to define such a structure?
- All components are tightly connected, side effects are everywhere
- There is no “one-simple reason” explaining their performance (supposition)
- At least we know that we don't know

Idea behind glucose
A real experimental study of CDCL solvers

CDCL solvers are complex systems – Illustration



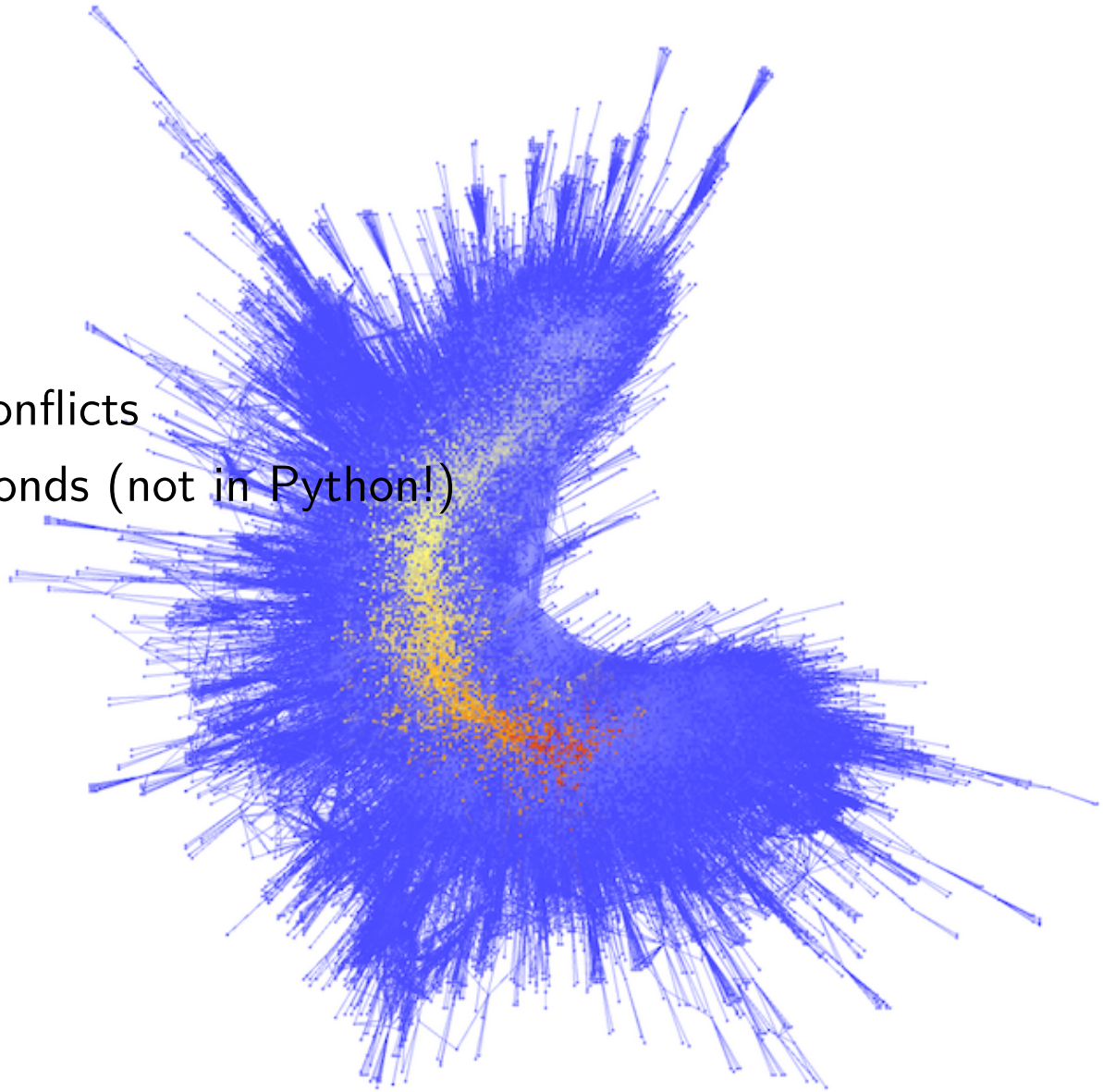
Example of a real conflict analysis:

- Many resolutions at each conflict
- Very reactive VSIDS (1/10s lifetime)

But: A clear structure behind!

A part of the research in SAT is hunting for the structure behind CDCL mechanisms

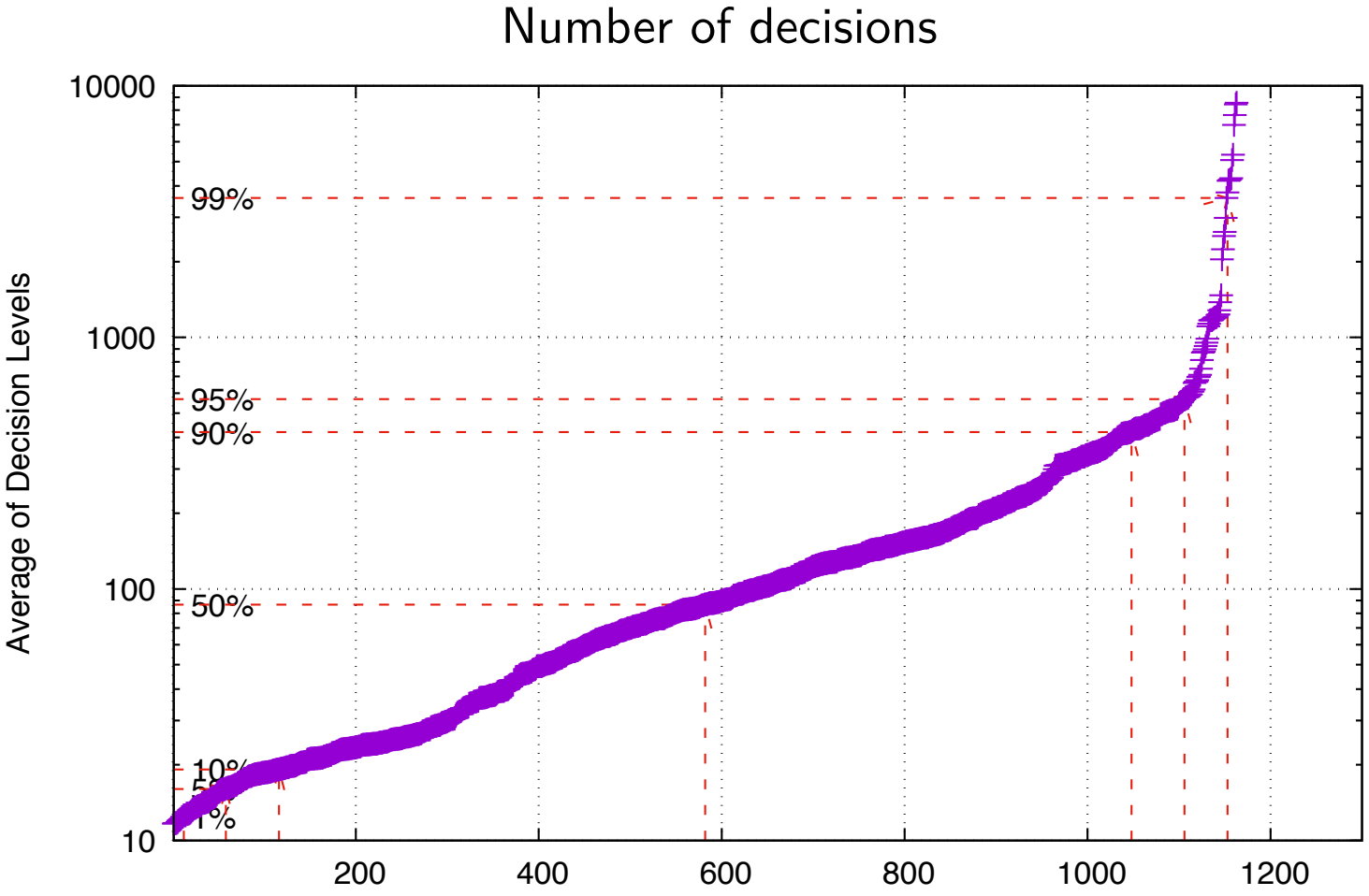
How to understand this (trivial) proof?



- Less than 70 000 conflicts
- Solved in a few seconds (not in Python!)
- A very dense proof
- Hard to understand

Outliers everywhere!

We try to understand CDCL solvers but all problems are distincts!



stats after 10,000 conflicts on 1164 “not easy” problems from all previous contests

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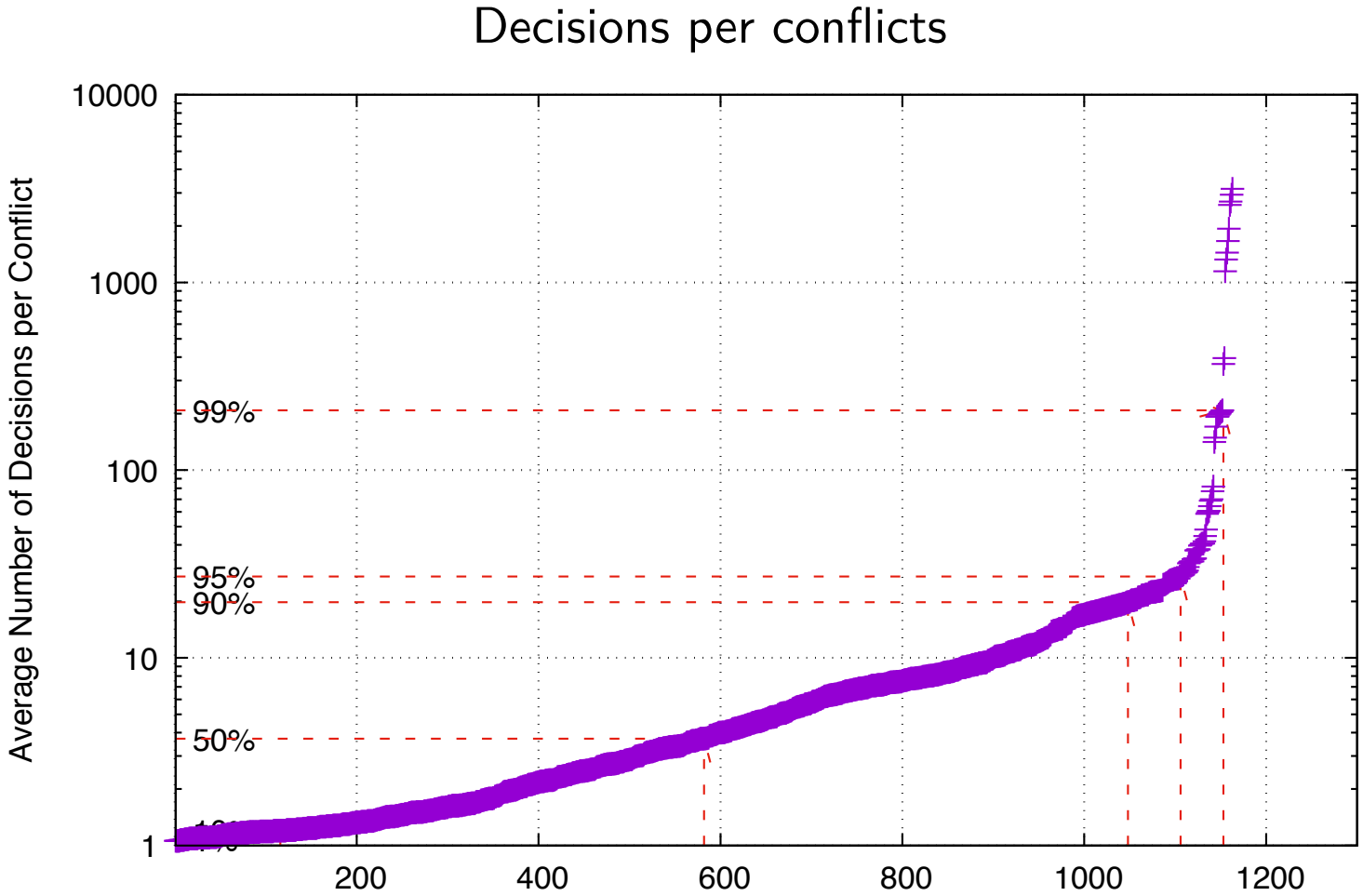

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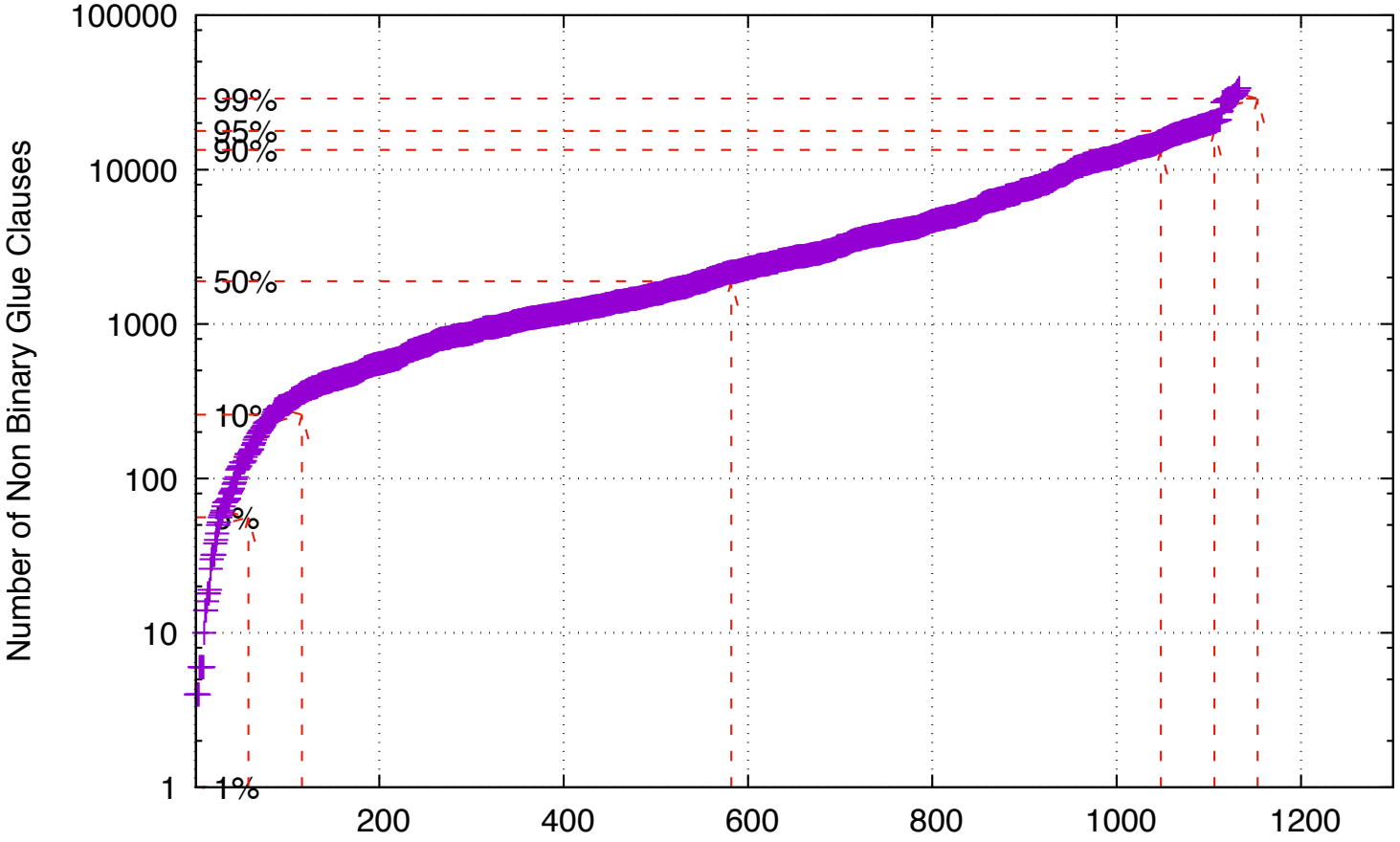
"True" (non binary) glue clauses



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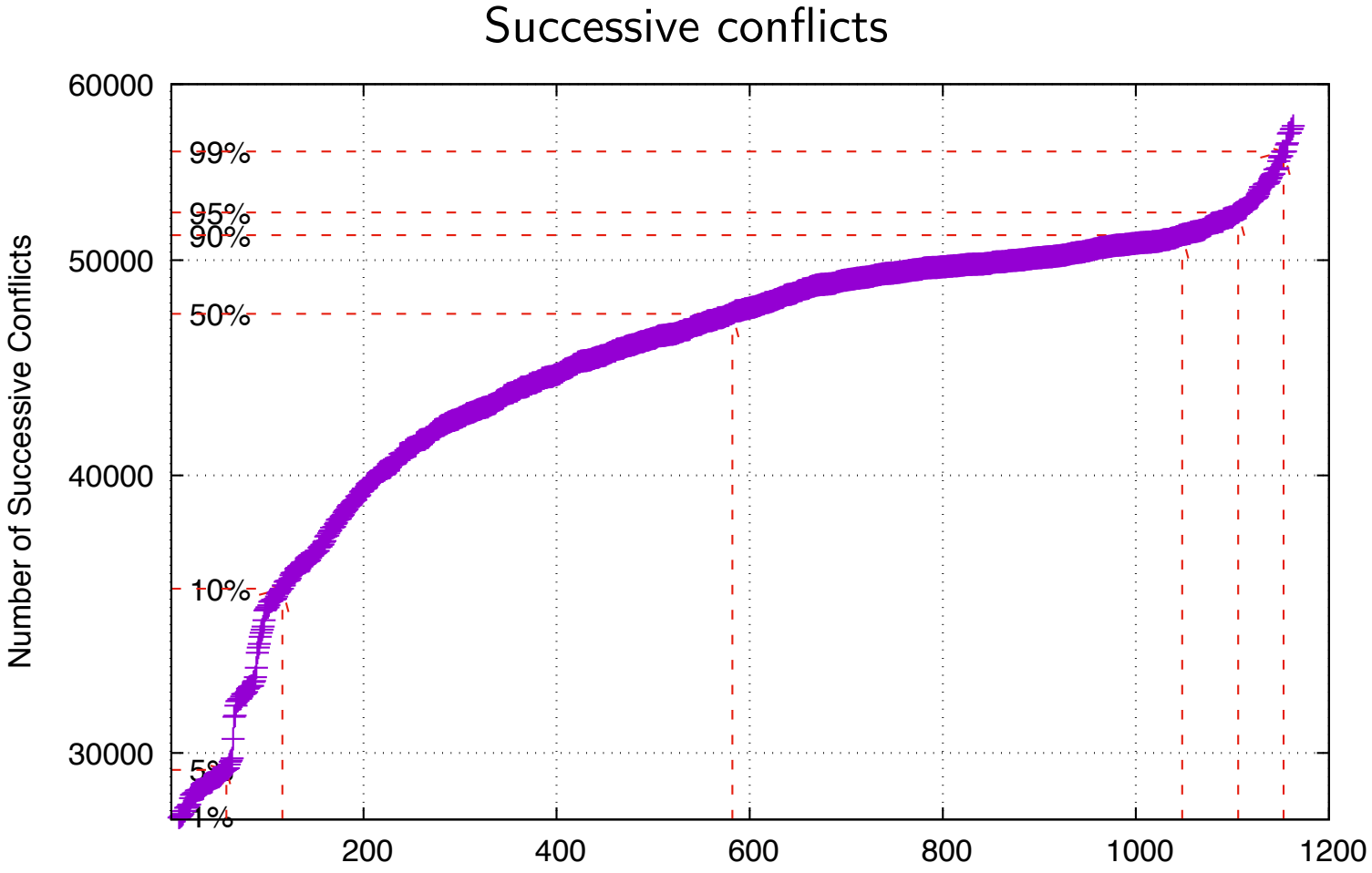

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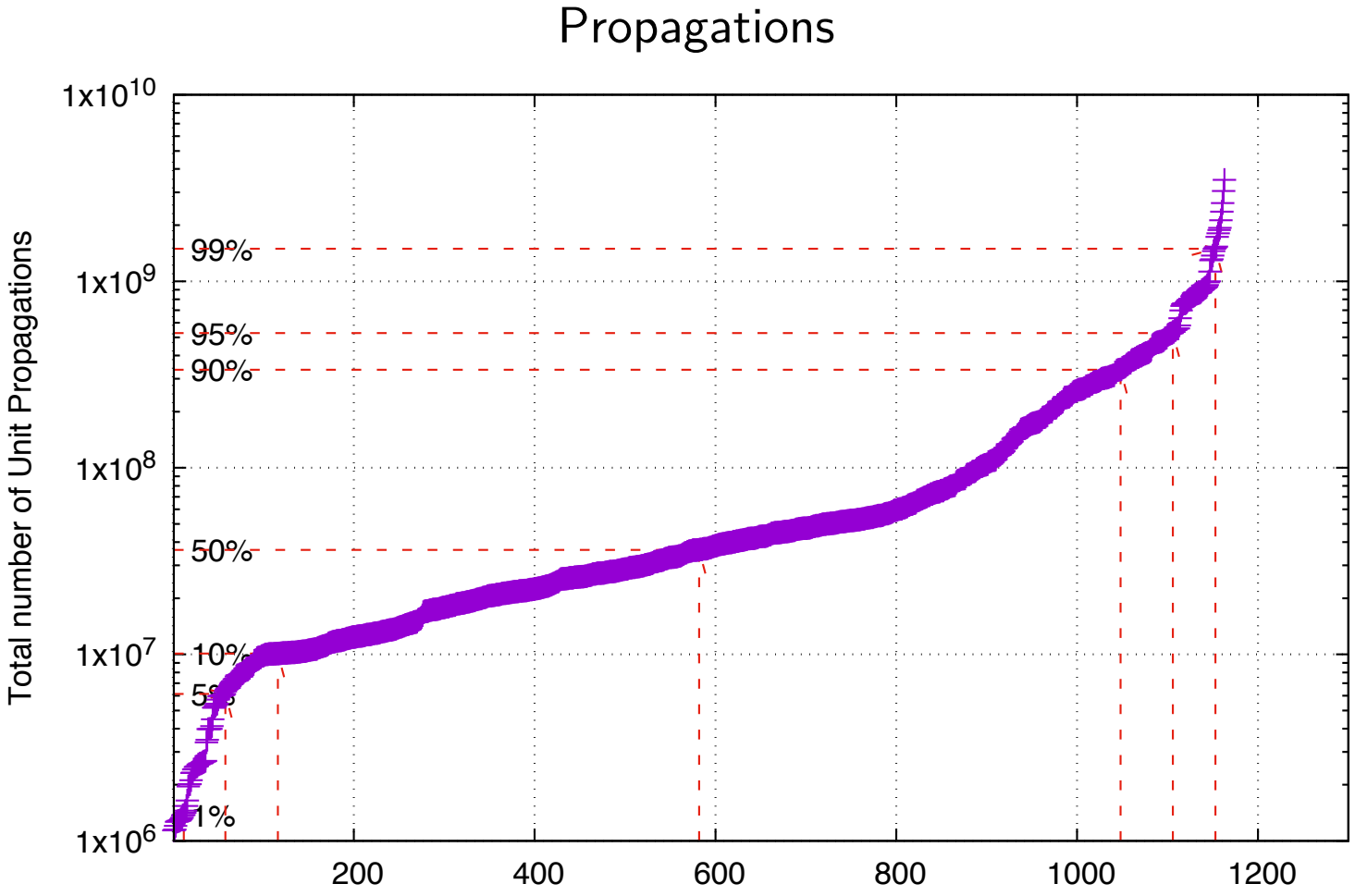


stats after 10,000 conflicts on 1164 “not easy” problems from all previous contests

When experimenting suggests that CDCL solvers are inefficient

Most of solver's time is spent in unit propagation

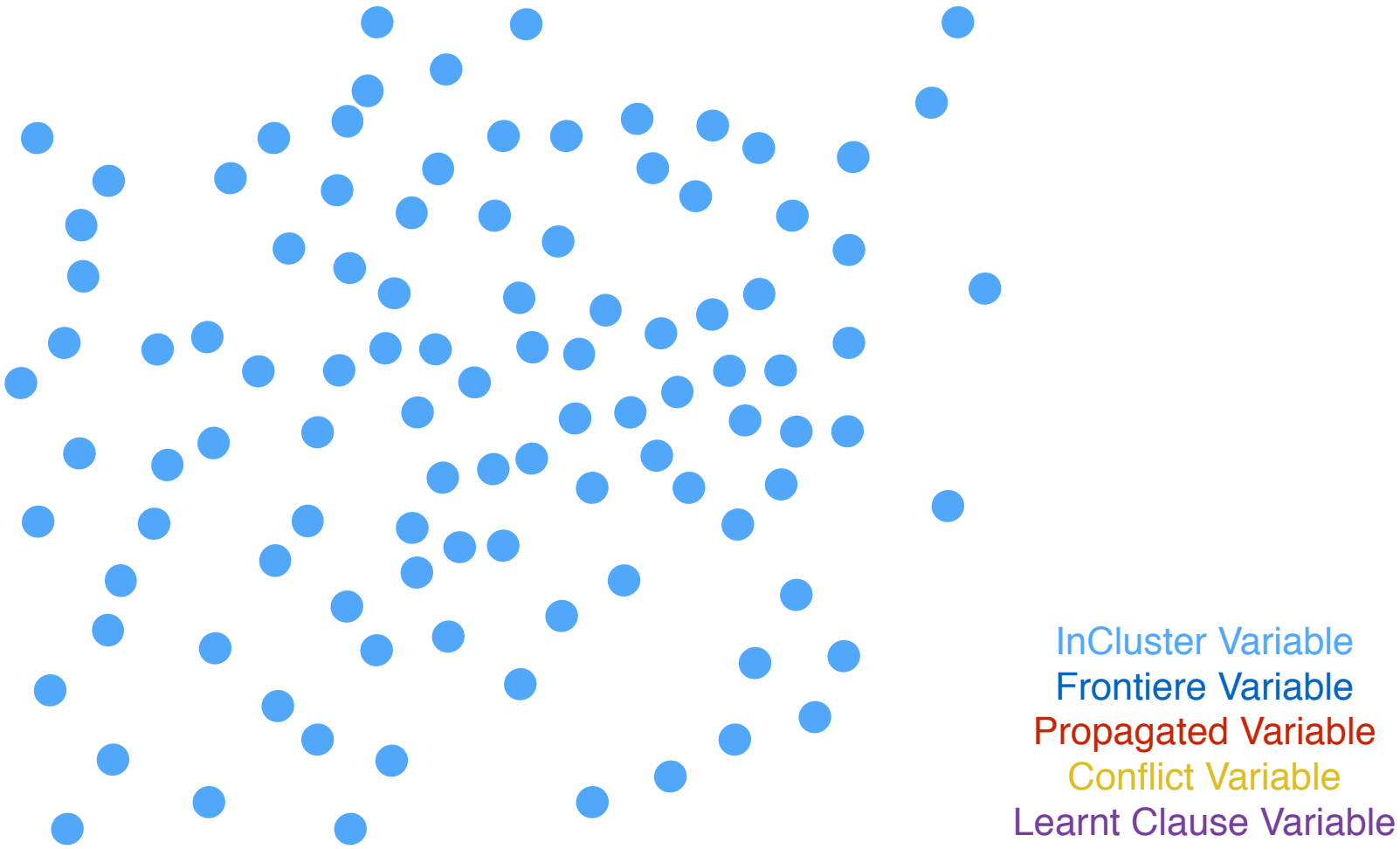
But, on UNSAT instances:

- Only 50% of generated clauses are useful for deriving the final contradiction
- Only 20% of unit propagation are used during conflict analysis

Only 10% of solver's time is useful for deriving the contradiction!

Today's Itinerary

(Possible) Illustration of learnt clause mechanism



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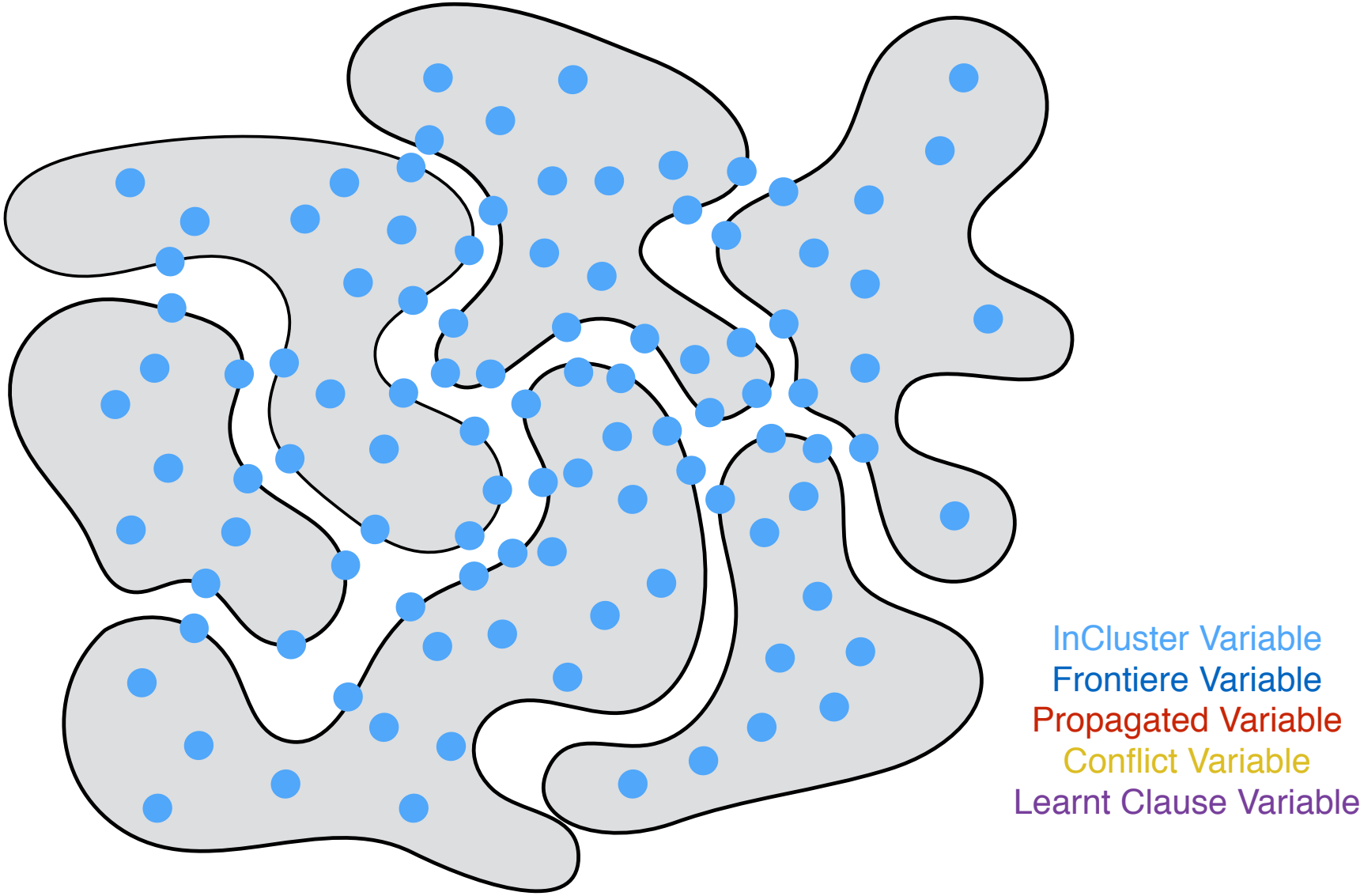
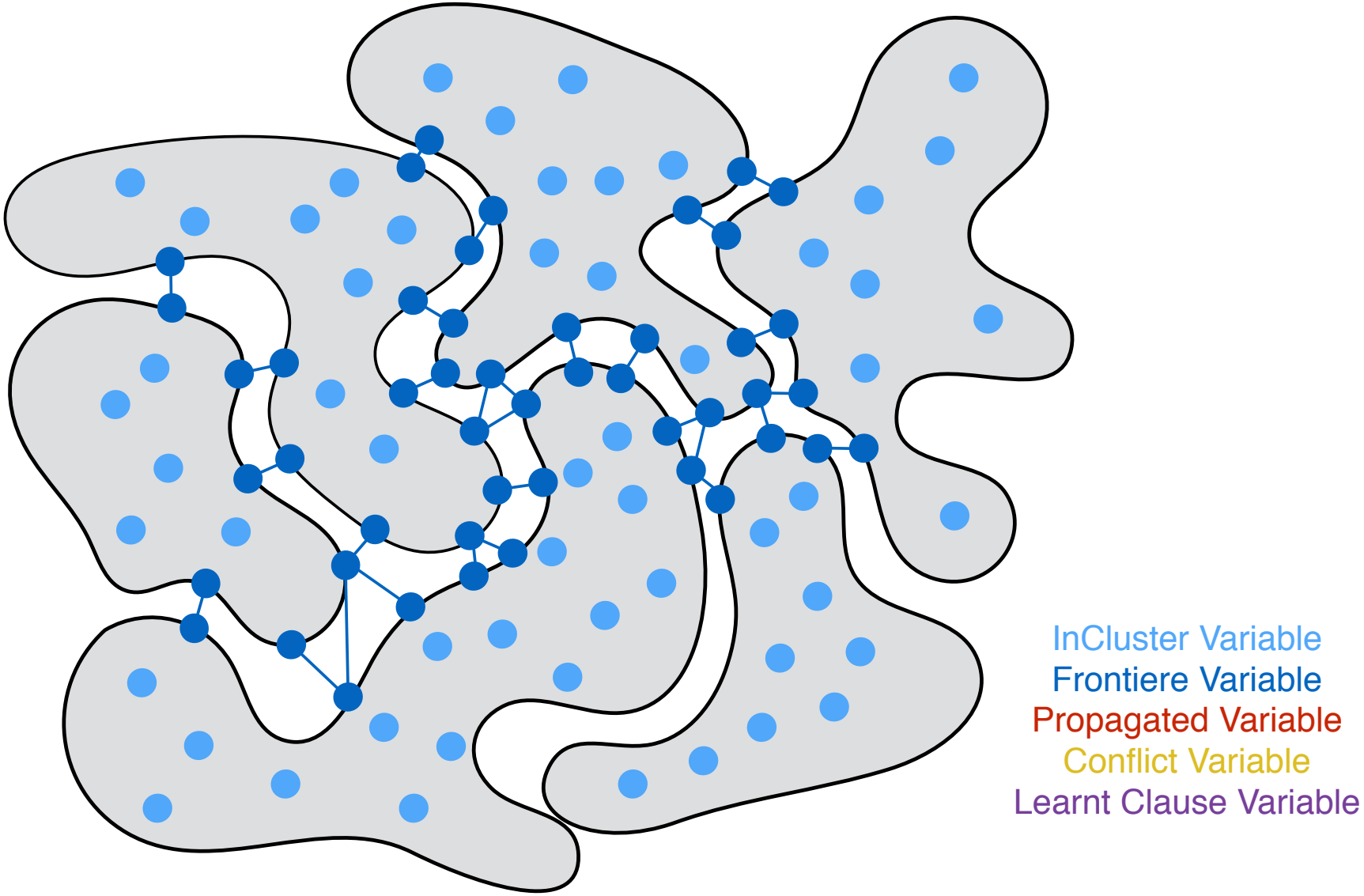
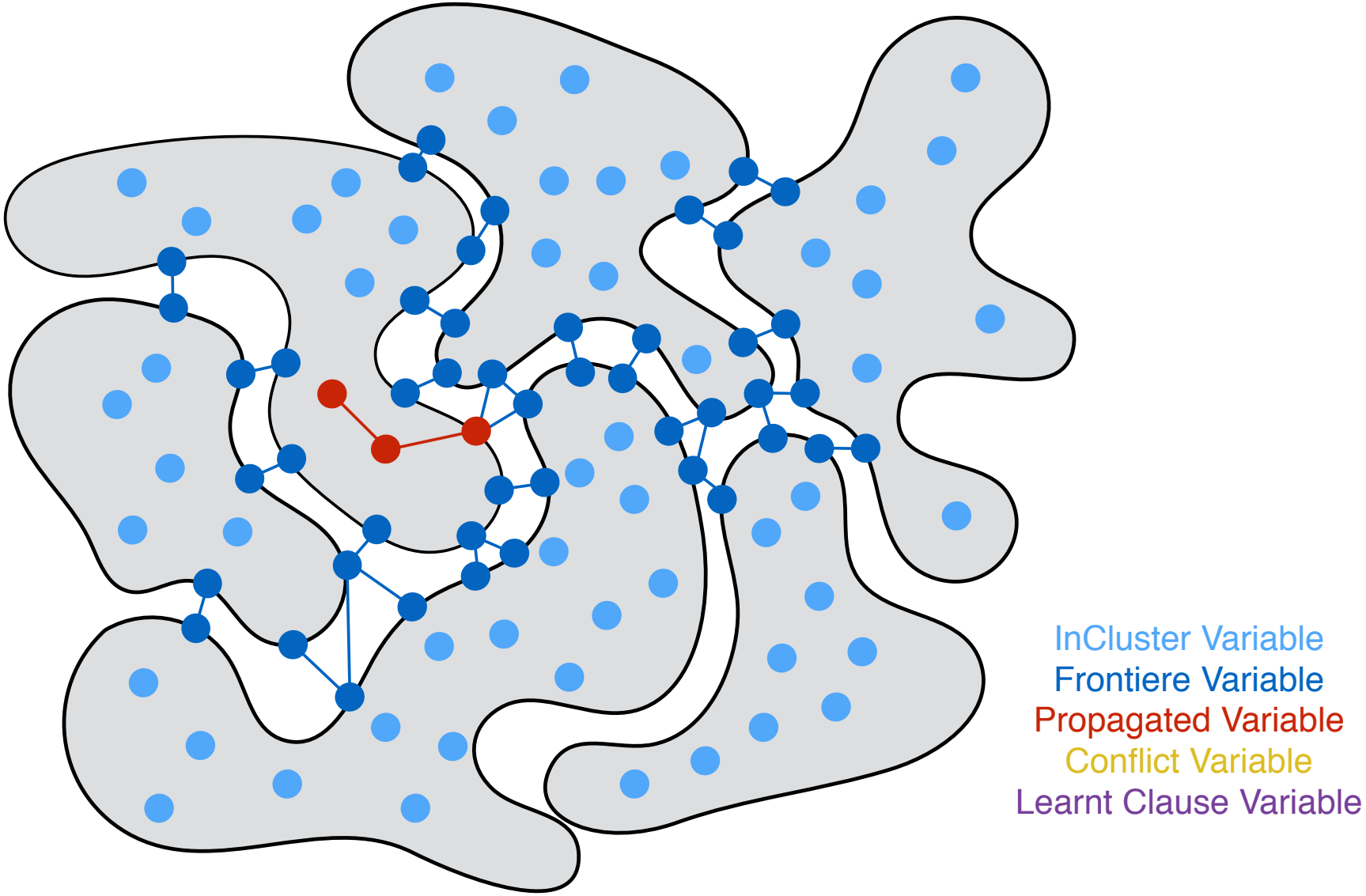


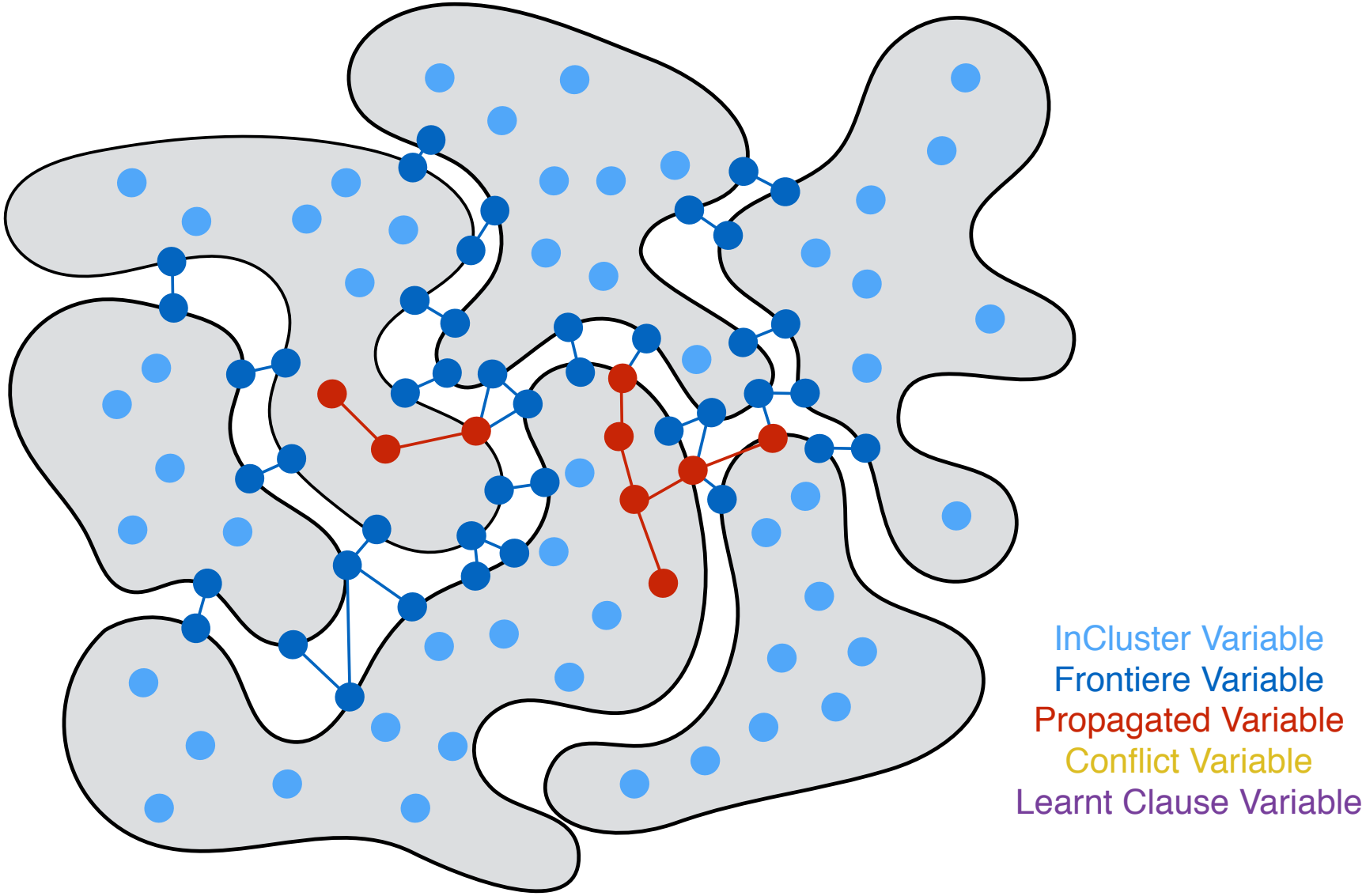
(Possible) Illustration of learnt clause mechanism



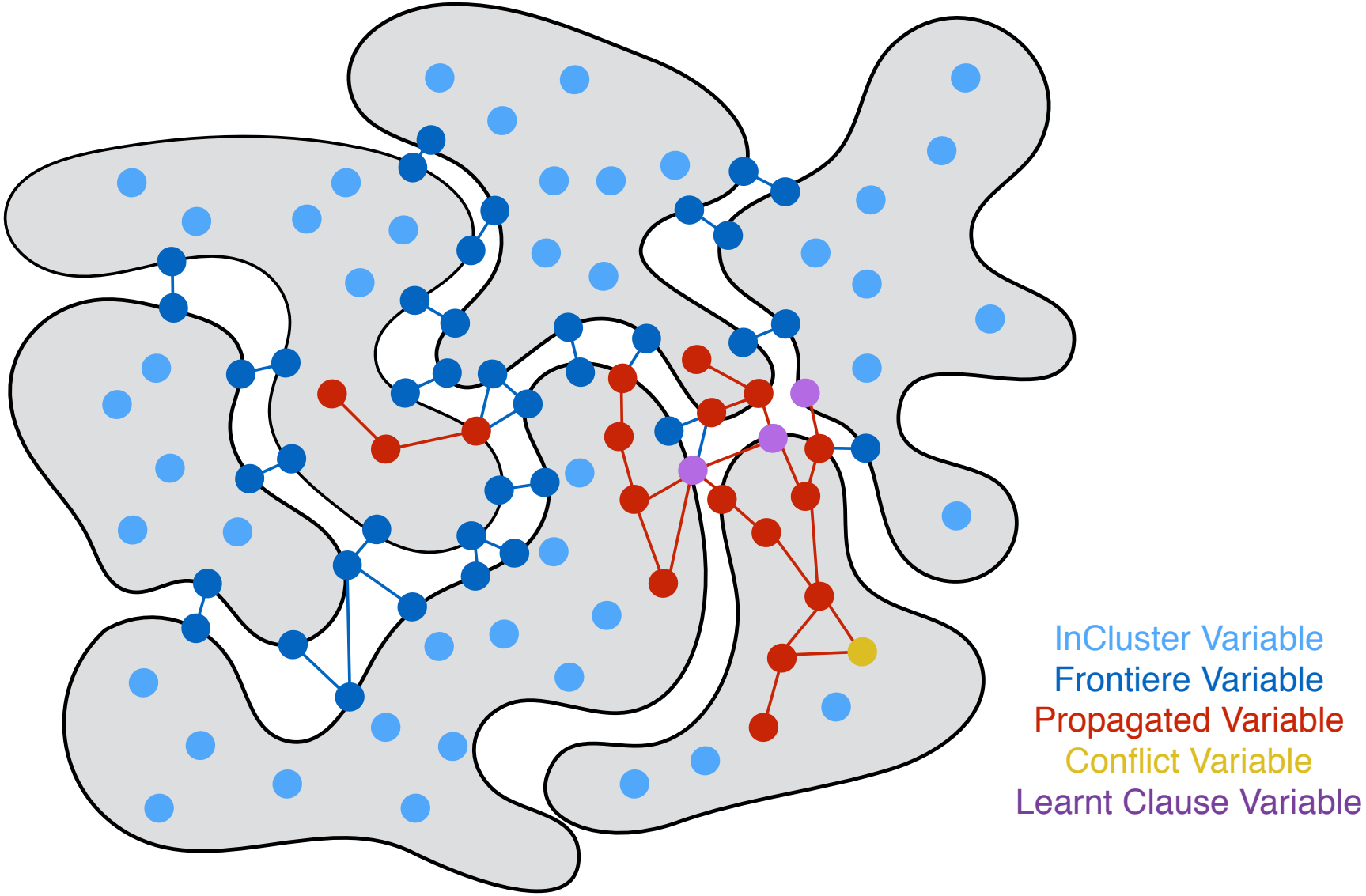
(Possible) Illustration of learnt clause mechanism



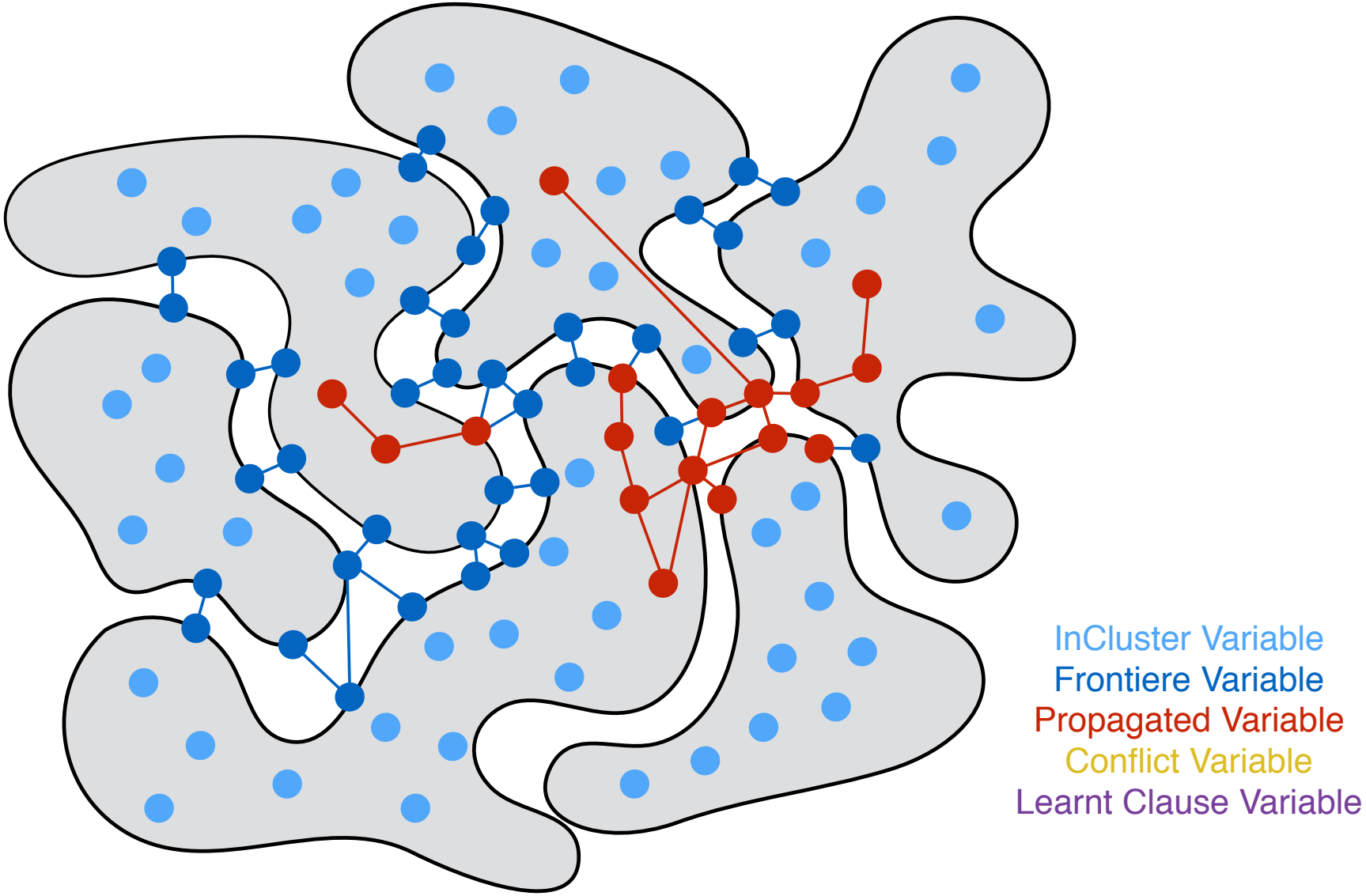
(Possible) Illustration of learnt clause mechanism



(Possible) Illustration of learnt clause mechanism



(Possible) Illustration of learnt clause mechanism



Conclusion

We now how to built an efficient SAT Solver but we can hardly explain their power

Experimenting is essential now in SAT research

LBD are related to communities (but may be more "semantic"

We (may) need to see CDCL has clauses producers, not branching algorithms
(much more difficult to handle, but closer to reality)

What about modelizing SAT solvers?